Transcendental Brauer-Manin Obstruction for a Diagonal Quartic Surface

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Synopsis

We study rational solutions to the polynomial equation with integral coefficients $f := x_0^4 + 3x_1^4 - 4x_2^4 - 9x_3^4 = 0$. In particular we prove that any integral solution $[y_0 : y_1 : y_2 : y_3]$ satisfies 3 divides the product y_1y_3 using Brauer groups.

In projective 3-space f defines a diagonal quartic surface X. Such a surface belongs to the class of K3 surfaces, whose geometry is well understood, but whose arithmetic properties are currently researched. A first indication about the arithmetic complexity of an arbitrary variety over a number field is given by its adelic points. This is part of the so called local-to-global theory. To a variety we have associated Brauer groups. The elements of those groups potentially give a refinement of the knowledge obtained from local-to-global methods called Brauer-Manin obstruction. It is yet unknown to which extent the Brauer-Manin obstruction rules the arithmetic of K3 surfaces, albeit some progress has been made in recent years.

First we introduce the necessary background in geometry and arithmetic to study X. Then we give an account of the theory of Brauer groups and the associated Brauer-Manin obstruction. These are well-known results but scattered in the literature. Finally we explicitly compute a transcendental 3-torsion Brauer element on a related surface inducing an element of $\mathrm{Br}(X)$, which explains a failure of weak approximation that manifests itself in $3|y_1y_3$. We use a computer algebra system in these computations. Although it was expected that transcendental 3-torsion Brauer elements are significant for an obstruction in this context, the explicit construction of such an example is a novelty.

Zusammenfassung

Wir untersuchen rationale Lösungen der polynomialen Gleichung mit ganzzahligen Koeffizienten $f := x_0^4 + 3x_1^4 - 4x_2^4 - 9x_3^4 = 0$. Wir zeigen mit Hilfe von Brauergruppen, dass jede ganzzahlige Lösung $[y_0 : y_1 : y_2 : y_3]$ die Teilbarkeitsrelation, dass 3 das Produkt y_1y_3 teilt, erfüllt.

Das Polynom f definiert eine diagonale quartische Fläche X im projektiven 3-Raum. Solch eine Fläche gehört zur Klasse der K3 Flächen. Ihre Geometrie ist wohlverstanden, ihre Arithmetik wird derzeit erforscht. Ein erstes Kriterium für die Existenz rationaler Punkte auf Varietäten, welche über dem rationalen Zahlkörper definiert sind, ist durch ihre adelischen Punkte gegeben, was zur sogenannten lokal-global-Theorie gehört. Einer Varietät sind gewisse Brauergruppen zugeordnet. Deren Elemente wiederum können potentiell eine Verfeinerung der Ergebnisse der lokal-global-Theorie vermitteln, was als Brauer-Manin-Obstruktion bezeichnet wird. Es ist nicht klar inwieweit die Brauer-Manin-Obstruktion schon das gesamte arithmetische Verhalten von K3 Flächen festlegt, obschon in den letzten Jahren Fortschritte erziehlt wurden.

Wir beginnen mit einer Einführung in die zur Untersuchung von X benötigten geometrischen und arithmetischen Grundlagen. Wir fahren fort mit einer Darlegung der Theorie der Brauergruppen und der Brauer-Manin-Obstruktion. Dies sind bekannte Resultate, die in der Literatur verstreut zu finden sind. Schließlich berechnen wir explizit ein transzendentes 3-Torsionselement der Brauergruppe einer zu X verwandten Fläche, das ein Element von $\mathrm{Br}(X)$ repräsentiert. Dabei wird auf die Benutzung eines Computeralgebrasystems zurückgegriffen. Dieses Brauerelement erklärt das Versagen von schwacher Approximation, welches sich in der Teilbarkeitsrelation $3|y_1y_3$ widerspiegelt. Die Signifikanz von transzendenten 3-Torsionselementen in Brauergruppen für eine solche Obstruktion war erwartet, die explizite Konstruktion eines solchen Beispiels ist eine Neuerung.

Das reicht, wenn du's morgen abend weißt.
Joanna Bauer, Bergen, 24.08.07

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Contents

CHAPTER 1

Introduction

The aim of this thesis is to investigate the arithmetic behavior of a diagonal quartic surface. Concretely we give an example of a transcendental 3-torsion element of the associated Brauer group that gives rise to an obstruction to weak approximation on this surface. We discuss all necessary background to this end and give an account for the explicit calculations and the algorithms involved. We also discuss a related topic, namely an effective method in Galois cohomology in the appendix.

1.1. Philosophy

Today the very first thing children learn in school is - next to the alphabet - counting in the integers. This may be the reason why layman equate mathematics with computing with numbers. But why do even most mathematicians consider the integers to be more important than any other number system? If one accepts the importance of (associative, unital, commutative) rings, that is, the importance of addition, subtraction and multiplication for numbers, the question may be rephrased: What makes the integers a special ring? Category theory tells us that there are two distinguished rings, namely the terminal ring and the initial ring. The terminal ring is the 0-ring which is undoubtably very important but rather boring and therefore often neglected. The initial ring is $\mathbb{Z}$, the integers.

Number theory is concerned with the properties of the ring of integers. Algebra is concerned with combinations of addition, subtraction and multiplication and most often only with combinations of finitely many of these like polynomials and equations made up by them. One of the most basic problems in the intersection of algebra and number theory is to study diophantine equations:

Problem 1.1 (Dioph Eq). Let $n, m \in \mathbb{N}_0$ and $F := (f_1 := \sum_{j\in\mathbb{N}_0^n} a_{1,j}x^j, \dots, f_m := \sum_{j\in\mathbb{N}_0^n} a_{m,j}x^j) \in \mathbb{Z}[x_1, \dots, x_n]$ where the x_i are indeterminates. What are the properties of the integral solution set

$$\mathbb{L}_{F/\mathbb{Z}} := \mathbb{L}_F := \{\underline{y} \in \mathbb{Z}^n : F(\underline{y}) = \underline{0} \in \mathbb{Z}^m\}?$$

A more concrete version of this problem is the tenth on the famous list of Hilbert's problems, which reads at least morally:

Problem 1.2 (Hilb 10th). Is there a (Turing-) algorithm that decides on arbitrary input n, m, F as in problem 1.1, if $\mathbb{L}_F = \emptyset$?

Unfortunately there cannot exist such an algorithm as was worked out by Matiyasevich et al. (see [**92**]) from the 40s to the 70s of the last century. A way to cope with this and the overall hardness of the problem is to try to restrict the allowed input. Over the centuries this idea was pursued. The first idea is to put restrictions on the algebraic form of F like degree, number of variables, number of equations, number of terms or homogenity. This was successful to some extent. During these investigations, mathematicians discovered that in many cases algebraic restrictions forced geometric restrictions, and that really these geometric restrictions are responsible for the success.

How does geometry enter? Instead of $\mathbb{L}_{F/\mathbb{Z}}$ one may consider

$$\mathbb{L}_{F/\mathbb{C}} := \{\underline{y} \in \mathbb{C}^n : F(\underline{y}) = \underline{0} \in \mathbb{C}^m\},$$

which is possible in a well-defined way since the integers are the initial ring and one may therefore consider the integral coefficients of the polynomials in F uniquely as complex numbers giving rise to complex polynomials. These sets $\mathbb{L}_{F/\mathbb{C}}$ are subsets of the affine space $\mathbb{C}^n$, which is the prototype of a complex manifold. If F meets certain smoothness conditions, $\mathbb{L}_{F/\mathbb{C}}$ turns out to be a complex (holomorphic) submanifold with induced structure. The behavior of $\mathbb{L}_{F/\mathbb{Z}}$ (finiteness, "parametrization") was discovered to be linked to geometric invariants of $\mathbb{L}_{F/\mathbb{C}}$ (dimension, "genus"). Over time mathematicians came up with geometric objects that are more suited to algebra and that can deal, e.g., with non-smooth situations. Today very often schemes and abstract varieties are used in place of manifolds.

This is the idea of arithmetic geometry: to study problems from (algebraic) number theory using methods of (algebraic) geometry under the motto (from [**67**])

Geometry Determines Arithmetic.

In this thesis we work based on the ZFC axiom system. The occasional use of categories of course pose a foundational problem. Some successful approaches (NBG or Grothendieck universes, see [**4**, I.1.]) to basic versions of this problem requiring at most mild further consistency assumptions. Those would be enough for our purposes. However the obfuscating technicalities related to these theories are not the main point of this work, and we naively act as if this problem would not occur. We are however exemplary careful in the definition of the Brauer groups for fields to avoid mystification of this problem.

We try to use "=" in mathematical formulas only for actual set theoretic equality and indicate the existence of an isomorphism in the appropriate category suggested by the context by the symbol "$\cong$". This applies even to canonical isomorphisms, i.e., for isomorphisms being the application of a (not necessarily explicitly stated) natural transformation between two functors to an object. When we abandon this strict policy for certain exceptions we indicate it explicitly.

By ring we mean an associative commutative and unital ring. When R is a ring in this sense, then an R-algebra A is an associative and unital R-algebra, i.e., A is an associative unital but not necessarily commutative ring with a homomorphism $R \to A$ in the category of associative unital but not necessarily commutative rings. We assume the analogous statements for sheaves of rings, e.g., for $\mathcal{O}(X)$ for a scheme X, and for a sheaf of $\mathcal{O}(X)$-algebras $\mathcal{A}$.

1.2. Overview

From a geometric point of view projective varieties have much nicer properties than arbitrary varieties defined by a general F as in problem 1.1. F yields a projective varieties if the polynomials of F are homogeneous. As a consequence, since rescaling is intrinsic in projective structures and therefore division by arbitrary elements of the base ring is implicitly possible, we may assume to work with the rationals rather than the integers. Using $\sim$ for the usual relation for projective points our restricted problem now is:

Problem 1.3. Define for $n, m \in \mathbb{N}_0$ and $F \in \mathbb{Z}[x_0, \ldots, x_n]$ homogeneous the set

$$\begin{aligned}\mathbb{P}_{F/\mathbb{Q}} := \{[\underline{y}] \in (\mathbb{Q}^{n+1} \setminus \{\underline{0}\})/\sim_{\mathbb{Q}} \colon F(\underline{y}) = \underline{0} \in \mathbb{Q}^m\} \cong \\ \{[\underline{y}] \in (\mathbb{Z}^{n+1} \setminus \{0\})/\sim_{\mathbb{Z}} \colon F(\underline{y}) = \underline{0} \in \mathbb{Z}^m\}.\end{aligned}$$

What are "good sets" (informally called classes) $\mathbf{X}$ of triples (m, n, F) such that the behavior of the associated $\mathbb{P}_{F/\mathbb{Q}}$ can be described to a satisfying amount? Which properties can be used to define such classes?

Historically the following properties have proven useful: smoothness, integrality (= reduced and irreducible), dimension (of $\mathbb{P}_{F/\mathbb{Q}}$) and codimension (of $\mathbb{P}_{F/\mathbb{Q}} \subset \mathbb{P}^n_{\mathbb{Q}}$), degree (of $\mathbb{P}_{F/\mathbb{Q}} \subset \mathbb{P}^n_{\mathbb{Q}}$), genus, Kodaira dimension and related classification results. After the trivial cases (dimension or codimension is 0) the best understood cases are those of curves (dimension is 1) and of hypersurfaces (codimension is 1).

For smooth curves that have at least one rational point, i.e., $\mathbb{P}_{F/\mathbb{Q}} \neq \emptyset$, there is a final qualitative description of the arithmetic behavior. It consists of the parametrization result for genus 0, the Mordell-Weil theorem for genus 1, stating the existence of a polynomial finitely generated abelian group structure on $\mathbb{P}_{F/\mathbb{Q}}$, and Faltings theorem (= Mordell conjecture) for genus ≥ 2, declaring the finiteness of $\mathbb{P}_{F/\mathbb{Q}}$. Unfortunately quantitative or algorithmically effective variants are only known for genus 0. For genus 1 the conjecture of Birch and Swinnerton-Dyer, which is part of the seven Millenium problems and can be considered extremly hard, predicts the quantitative behavior. In the remaining case there are heuristics such as Chabauty's method (= Chabauty-Coleman method), which are successful in many lucky cases, but nothing as definitive as in the other cases.

The next big class that is considered are (smooth) surfaces (dimension 2). There is a classification that is explained with the rest of the above notions in greater detail in chapter 2. For example the (geometrically) rational surfaces contain a subclass called del Pezzo surfaces for which the quantitative behavior is predicted by the Manin conjecture (sometimes called Batyrev-Manin conjecture). For abelian surfaces, i.e., the higher dimensional analogs of elliptic curves, that is smooth curves of genus 1 with at least one rational point, the (generalization of the) Mordell-Weil theorem holds.

One of the remaining classes are K3 surfaces. The K3 surfaces coming from polynomial equations as above fall in one of infinitely many (not mutually exclusive) classes, one of which is the quartic surfaces, i.e., smooth surfaces which can be embedded as hypersurfaces of degree 4 in projective 3-space, arguably the most important due to their simplicity. Diagonal quartics on their part are the simplest quartic surfaces, making them the ideal test case for K3 surfaces amenable to explicit computations. The main result of this thesis concerns such a surface, explicitly the surface defined by the polynomial

$$x^4 + 3y^4 - 4z^4 - 9w^4 \in \mathbb{Z}[x, y, z, w].$$

Next to basic results from geometry we also discuss the Hasse principle, weak approximation and the background for it like adels in chapter 2.

In the chapter on Brauer groups we define them and compile all the results from the literature necessary for understanding later computations. This includes some spectral sequences, the relation between the Brauer group and the cohomological Brauer group, connections to Galois cohomology, local class field theory and Hasse reciprocity (= theorem of Albert, Brauer, Hasse, Noether), the different parts of the Brauer group of a surface (constant, algebraic and transcendental), a description for the Brauer-Manin obstruction and an overview of results and developments in Brauer-Manin obstruction to surfaces in general and diagonal quartics in particular.

In the final chapter we give an explicit example of Brauer-Manin obstruction to the diagonal quartic defined above. We construct an explicit non-trivial transcendental 3-torsion element, giving rise to a Brauer-Manin obstruction to weak approximation over the rationals. We first give an outline of the construction which is not necessary in the order of how one comes up with such a Brauer group element, but is better suited for exposition. In following sections we explain some technical results needed for the application. Then we give a short outline how one could construct transcendental 2-torsion elements on diagonal quartics with these methods. Since transcendental 2-torsion elements are covered by Ieronymou in [**69**], we do not carry out the calculations. The methods of Ieronymou and those of this thesis are different which makes the outline worth wile.

In the first appendix we give a solution to a problem in Galois cohomology, which is related to effective Brauer-Manin obstruction and was produced during the making of this thesis. The result itself is probably well-known, but an explicit account of it was given only recently by Fieker in [**40**] using different methods. We also give explicit formulas.

In the second appendix we include code used in the computations of chapter 4. It is commented to some extent and consists of scripts written for `MAGMA`.

In the last appendix we give a list of integral solutions to the equation defined by the above polynomial with small coordinates.

CHAPTER 2

General Arithmetic Geometry

In this chapter we introduce and explain the necessary notions and results from algebraic geometry and arithmetic geometry. We mostly clarify notions which are differently defined throughout the literature to avoid confusion and compile a series of results and techniques used later, which are not original at all.

The standard references for algebraic geometry are the book of Hartshorne [**61**] for a moderately thorough but concise introduction, Grothendieck's EGA [**55**], which is even 50 years after its publication the ultimate reference of the subject, and the book of Griffiths and Harris [**51**], which has a view towards complex analytic geometry.

A good general introduction to arithmetic geometry is [**67**] of Hindry and Silverman, especially for curves. Silverman's several books on elliptic curves require less or no knowledge of scheme theory and give a good impression of standard procedures in the subject (reduction mod p, heights, group varieties, etc.) but restrict to a special type of arithmetic variety. Liu's book [**87**] is an introduction to algebraic geometry but with arithmetic in mind; arithmetic models of varieties and "reduction mod p" are treated very thoroughly and general compared to other books on diophantine equations.

The above mentioned books [**61**] and [**51**] also discuss special properties of surfaces. The standard reference is "Compact Complex Surfaces" [**9**].

An accesible proof of the Hasse-Minkowski theorem, the prototype of the Hasse principle (= local-to-global principle), can be found in Serre's [**114**], compact definitions of the adeles in [**90**] of Manin and Panschishkin or in the recent book of Bombieri and Gubler [**15**]. More detailed observations on adeles can be found in [**20**] edited by Cassels and Frölich, or in Weil's [**129**]. Building on this, a short and concise introduction to the general Hasse principle and weak approximation is contained in the book of Skorobogatov [**120**, 5.1.].

For the Weil bounds discussed in relation to local solubility see Poonen's book in preparation [**104**], which might also serve as a good introduction to general arithmetic geometry, and for a more traditional introductory coverage of the Weil conjectures see notes of Ile [**71**]. The Weil bounds were established in Lang's and Weil's article [**85**].

The miscellenea section collects several results on fibrations over number fields and diagonal quartics distributed in the existing literature.

2.1. Algebraic Geometry and Classification of Surfaces

We begin with fixing some basic notation.

DEFINITION 2.1. Let k be a field. A k-variety is a geometrically integral[1] separated scheme X of finite type over $\mathrm{Spec}(k)$.
A variety is a k-variety for some field k.
The function field $k(X)$ of a variety X is defined to be the residue field at the unique generic point (see [**61**, II Exercise 2.9.]) $x \in X$, which in this case equals the local ring at x, i.e., $k(X) := k(x) = \mathcal{O}_{X,x}$.
Let k be a number field and $\mathfrak{o}_k$ its ring of integers, i.e., the integral closure of $\mathbb{Z}$ in K. Let R be a ring obtained from $\mathfrak{o}_k$ by inverting finitely many elements. We call an integral separated flat R-scheme of finite type an arithmetic variety.

DEFINITION 2.2. A curve is a 1-dimensional variety, a surface is a 2-dimensional variety. k-curves and k-surfaces for a field k are defined analogously.

DEFINITION 2.3. Let k be a field, K/k a field extension and X a k-scheme. Then define $X_K := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$.
Furthermore define $X^{sep} := X_{k^{sep}}$ and $\overline{X} := X^{alg} := X_{k^{alg}}$, where we implicitly fixed an algebraic and a compatible separable closure for k.

DEFINITION 2.4. The residual characteristics of a scheme X are the set of characteristics of the residue fields of the points of X, and as such are a subset of $\mathbb{N}_0$. A scheme of characteristic 0 is a scheme whose residual characteristics are $\{0\}$.

DEFINITION 2.5. Let X and Y be schemes. The set of Y-points (Y-rational points) of X is defined as $X(Y) := \mathrm{Hom}_{\mathbf{Sch}}(Y, X)$. If $Y = \mathrm{Spec}(R)$ is an affine scheme where R is a ring, we write $X(R) := X(\mathrm{Spec}(R))$ and speak of the R-points (R-rational points) of X. If k is a field and X is a k-scheme, we call $X(k)$ simply the set of rational points.
In contrast the points of X without any further attribution still refer to the elements of the set underlying the topological space which constitutes X.

REMARK 2.6. There is also a relative notion of rational points, which in general differs from the absolute notion.
Let e.g. $k := \mathbb{Q}(\sqrt[3]{2})$ and $K := \mathbb{Q}(\sqrt[3]{2}, i)$. There are 3 different field homomorphisms $\alpha_j : k \hookrightarrow K$ for $j \in \{0, 1, 2\}$ uniquely determined by $\alpha_j(\sqrt[3]{2}) = i^j \cdot \sqrt[3]{2}$. Set $X := \mathrm{Spec}(k), Y := \mathrm{Spec}(K)$ and let $\beta_j : Y \to X$ be the morphisms induced by α_j. Both X, Y are k-schemes via the morphisms id_X, β_0, but only β_0 is a morphism of k-schemes, while β_1, β_2 are not.
Thus $\mathrm{Hom}_{\mathbf{Sch}}(Y, X) = \{\beta_0, \beta_1, \beta_2\} \neq \{\beta_0\} = \mathrm{Hom}_{k-\mathbf{Sch}}(Y, X)$.

DEFINITION 2.7. Let $f : X \to Y$ be a morphism of schemes. It is called connected, if and only if all its fibers are connected, i.e.,

$$\forall y \in Y : f^{-1}(\{y\}) \text{ is connected as a topological space.}$$

[1] i.e., $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{alg})$ is integral

REMARK 2.8. A scheme is called regular, if and only if all the stalks of the structure sheaf are regular local rings. If all these rings remain regular after tensoring with the algebraic closure of the respective residue field, it is called geometrically regular. An S-scheme X is called smooth, if its structure morphism is smooth. This holds in particular for k-varieties. For k-varieties with k perfect all three notions coincide (see [**104**, 3.5.5.]).
Some authors call a smooth k-variety non-singular and singular otherwise; sometimes $k = k^{alg}$ is required. Since this notion is not used consistently in the literature, we avoid it. The terminology of singular fibers for a fibration is introduced below, and the singular locus means the topological subspace (endowed with reduced induced subscheme structure when in doubt) of non-regular points. A singularity is a non-regular point.

REMARK 2.9. Using Chow's Lemma (see [**61**, II Exercise 4.10.] or more generally [**55**, II Théorèm (5.6.1)]), one could generalize many of the notions and results concerning projective schemes to proper schemes.

2.1.1. Birational Morphisms and Birational Maps

DEFINITION 2.10. A morphism between two schemes each having only finitely many irreducible components $f : X \to Y$ is called birational (see [**55**, IV (6.15.4)]), if and only if f induces a bijection between the set of points of X maximal with respect to inclusion between the respective topological closures of the points and the corresponding set for Y, and for any maximal $y \in Y$ the induced morphism of the residue fields $k(y) \to k(f^{-1}(y))$ is an isomorphism of fields.
Two schemes X, Y as above are called birational (birationally equivalent), if there exists a finite diagram of birational morphism (some of which may be chosen to be the identity morphism id) like this:

$$X \xrightarrow{f_1} Y_1 \xleftarrow{g_1} X_1 \xrightarrow{f_2} Y_2 \xleftarrow{g_2} X_2 \ldots \xrightarrow{f_n} Y_n \xleftarrow{g_n} X_n \xrightarrow{f_{n+1}} Y.$$

Let P be a property of morphisms of schemes. We say that X and Y are birational via morphisms satisfying property P, if there is a diagram as before such that the f_i and g_j have property P.

REMARK 2.11. We use the property birational mostly for morphisms of varieties over a common ground field. In this case this property is characterized by $f(x) = y$, where x and y are the respective generic points, and that the induced morphisms between function fields $k(y) = k(Y) \to k(X) = k(x)$ is an isomorphism.
We use the general definition when discussing proposition 3.78, which compiles some results of Grothendieck on the birational invariance of the Brauer group.
Any chain of birational morphisms as above with additionally all schemes locally noetherian and all morphisms of finite type can be shortened by 2.14 (5) to $X \xleftarrow{g} M \xrightarrow{f} Y$ with f, g open immersions. If we demand the morphisms to have other certain properties, the morphisms in the just constructed shortened chain may not have this property.

REMARK 2.12. In the literature birational is often defined via birational maps rather than via birational morphisms, the later demanding to restrict to schemes having only finitely many irreducible components. These two notions are equivalent under this and some additional finiteness condition, as we see shortly. The possibility to demand extra properties of birational morphisms giving the birational equivalence is useful later, so we prefer this definition. Since we work mostly with varieties, the slight restrictions we made are of no importance to us.

Since we are not aware of a good reference in the literature for the equivalence of the two alternative approaches to birational equivalence, we work it out explicitly.

DEFINITION 2.13. A rational map $f : X \dashrightarrow Y$ between two schemes X and Y is an equivalence class of pairs $(U, f_U : U \to Y)$ where U is an open dense subscheme of X, and f_U is a morphism of schemes. Two such pairs (U, f_U) and (V, f_V) are equivalent, if and only if the restrictions to the (open dense) intersection $U \cap V$ agree, i.e., $f_U\mid_{U\cap V} = f_V\mid_{U\cap V}$ (see [**55**, I. Definition (7.1.2)]).
A rational map $f : X \dashrightarrow Y$ represented by some (U, f_U) is called dominant, if and only if f_U is dominant, i.e., $f_U(U)$ is dense (cf. [**55**, I. Definition (2.2.6)]). It is a straightforward exercise in topology that this is well defined independent of the choice of (U, f_U).
Let $f : X \dashrightarrow Y, g : Y \dashrightarrow Z$ be rational maps between schemes representable by $(U, f_U), (V, g_V)$ such that $f_U^{-1}(V)$ is open dense in X (e.g., when f is dominant). Then we call f composable with g and $(U \cap f_U^{-1}(V), g_V \circ f_U\mid_{U\cap f_U^{-1}(V)})$ represents a rational map $g \circ f : X \dashrightarrow Z$ which is called the composition of f and g.
A rational map $f : X \dashrightarrow Y$ is a birational map, if and only if there is another rational map $g : Y \dashrightarrow X$ such that they are composable both ways and $g \circ f$ is represented by (X, id_X) and $f \circ g$ is represented by (Y, id_Y) ([**61**, I.4.]). In this case we call $f^{-1} := g$ the inverse map of f.

PROPOSITION 2.14. (1) *A rational map $f : X \dashrightarrow Y$ is birational if and only if it can be represented by some (U, f_U) such that f_U is a dominant open immersion, if and only if it can be represented by some (W, f_W) inducing an isomorphism $\hat{f}_W : W \xrightarrow{\sim} f_W(W)$ between open dense subschemes.*
(2) *Let $f : X \to Y$ be a morphism between schemes with only finitely many irreducible components. f is a birational morphism, if (X, f) represents a birational map, and the converse holds, if additionally f is of finite type and Y (hence X) is locally noetherian.*
(3) *Two birational maps $f : X \dashrightarrow Y, g : Y \dashrightarrow Z$ are composable and the composition is again a birational map. Furthermore $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$.*
(4) *Dominant open immersions and in particular isomorphisms between schemes with only finitely many irreducible components are birational morphisms.*
(5) *Two schemes with only finitely many irreducible components X and Y are birational, if there is a birational map $f : X \dashrightarrow Y$. The converse holds, if additionally X and Y are locally noetherian and they are birational via morphisms of finite type with locally noetherian domain and codomain.*

PROOF. If f is a birational map, then it can be represented by (U, f_U) such that there is a (V, g_V) representing some map $g : Y \dashrightarrow X$ with $g_V \circ f_U = \mathrm{id}_X |_U$, since we may restrict f_U if necessary. There are also representatives $(U', f_{U'}), (V', g_{V'})$ such that $f_{U'} \circ g_{V'} = \mathrm{id}_Y |_{V'}$. Without loss of generality (forming intersections $U \cap U', \ldots$ and induced restricted morphisms) we may assume $U = U'$ and $V = V'$. This means f_U induces an isomorphism onto an open dense subscheme V of Y, i.e., f_U is a dominant open immersion.
If $f_U : U \to Y$ is a dominant open immersion, then by definition it induces an isomorphism between the two open dense subschemes $U \subset X$ and $f_U(U) =: V \subset Y$. Thus there is an inverse isomorphism $g_V : V \to U$ and since U is open and dense it induces another dominant open immersion $\tilde{g}_V : V \to X$ giving rise to a rational map $g : Y \dashrightarrow X$. By dominance, f_U and $\tilde{g}_V$ are composable both ways, and since the construction of g involved inverse isomorphisms the compositions are the respective identities. This proves the first part.
Clearly (X, f) represents a rational map. If (X, f) represents a birational map, it can by the previous part of the proposition also be represented by some $(U, g := f |_U)$ restricting (X, f) such that g is an isomorphism of open dense subschemes. As such g induces a bijection between the set of maximal points of X and Y, and since g is a restriction of f, the same holds for f. g as an isomorphism induces isomorphisms on the stalks, in particular on the stalks of the maximal points, and again because it is a restriction of f the same holds for f.
If f is a birational morphism we have an induced bijection between the finitely many maximal points and isomorphisms on their stalks. By [**55**, I. Proposition (6.5.4) (ii)] f is a local isomorphism at the maximal points, i.e., there are open subsets U_x around each maximal point x such that $f |_{U_x}$ induces an isomorphism onto its openly embedded image. Taking the union U of all the U_x we see that f induces an isomorphism from U into an open $V \subset Y$. Since U, V contain the maximal points they are dense in X respectively Y and hence by the previously established facts (U, f) is a birational map, which concludes the prove of the second part.
By the first part of the proposition, birational maps are dominant, hence composable, and if we represent them by isomorphisms between open dense subschemes $f' : U' \to V', g' : V'' \to W'$, then by taking $h' := g' \circ f' |_{U' \cap f'^{-1}(V' \cap V'')}$, we again get an isomorphism between open dense subschemes, since intersections of open dense subschemes are again open dense. Thus the $g \circ f$ which is represented by h' is a birational map. The statement about the inverse of the composition is obvious.
The forth part is straightforward.
Let $f : X \dashrightarrow Y$ be a birational map. By the first part, we may establish a diagram

$$X \xleftarrow{\iota_X} U \xrightarrow{\theta} V \xrightarrow{\iota_Y} Y,$$

where ι_X, ι_Y are dominant open immersions, and θ is an isomorphism and as such are birational morphisms as well as the composition $\iota_Y \circ \theta$. The following diagram of birational morphisms exhibits X and Y as birational:

$$X \xrightarrow{\mathrm{id}_X} X \xleftarrow{\iota_X} U \xrightarrow{\iota_Y \circ \theta} Y.$$

Look at a chain of birational finite type morphisms of locally noetherian schemes:

$$X \xrightarrow{f_1} Y_1 \xleftarrow{g_1} X_1 \xrightarrow{f_2} Y_2 \xleftarrow{g_2} X_2 \ldots \xrightarrow{f_n} Y_n \xleftarrow{g_n} X_n \xrightarrow{f_{n+1}} Y.$$

Enumerate all of these morphisms by h_i. By the second part, each h_i gives rise to two birational maps, one is represented by $(\mathrm{Domain}(h_i), h_i)$, and the other is its inverse map. By the third part, we may compose them – alternating the induced rational map and the inverse of the induced map – and get a birational map as needed. □

Remark 2.15. The additional finiteness conditions in 2.14(2) and (5) are necessary to some extent. The inclusion of the generic point of any curve is a birational morphism which does not induce a birational map.

Remark 2.16. If $f : X \dashrightarrow Y$ is a rational map between a reduced scheme X and a separated scheme Y, then f has a unique representative (U, f_U) such that U is maximal (with respect to inclusion of open subschemes of X) – see [**55**, I. Proposition (7.2.2)]. U turns out to be the domain of definition of f, which may be defined for general rational maps [**55**, I. (7.2.1)].

Remark 2.17. Blowups of varieties are classical examples for birational morphism and they also appear prominently in the problem of desingularizing varieties (see Proposition 2.32 and for more details [**76**]). For surfaces over algebraically closed fields we know, that birational maps can be "factored" via blowups [**61**, V. Theorem 5.5.]. We will use these connections in an ad hoc way in the construction of our example in chapter 4.
Birational geometry in dimension greater than two is however much more involved.

2.1.2. Some Invariants of Curves and Surfaces

Before we come to results on the classification of low dimensional varieties, we introduce some invariants.

Definition 2.18. Let X be a k-scheme, and denote the sheaf of relative k-differentials by $\Omega_X := \Omega_{X/\operatorname{Spec}(k)}$. If $\dim(X) < \infty$, we can define the canonical sheaf $\omega_X := \bigwedge^{\dim(X)} \Omega_X$ (see [**61**, II. p.180]) which is an invertible sheaf, if X is a smooth k-variety.
If X is a smooth k-variety, then by [**61**, II.6.] isomorphism classes of invertible sheaves are in natural bijection with Cartier divisor classes and with Weil divisor classes. In this case, any Weil divisor whose class corresponds to the class of ω_X is called a canonical divisor and usually denoted by K_X. Its inverse $-K_X$ is then referred to as the anti-canonical divisor.

DEFINITION 2.19. Let k be a field and X a projective scheme over $\mathrm{Spec}(k)$ (hence proper, hence separated and of finite type and hence noetherian). Define the arithmetic genus of X (see [**61**, III. Exercise 5.3.]) to be

$$p_a(X) = (-1)^{\dim(X)}(\chi(\mathcal{O}_X)-1) = (-1)^{\dim(X)}(-1+\sum_{i=0}^{\dim(X)}(-1)^i \dim_{k-\mathrm{vs}} H^i_{zar}(X,\mathcal{O}_X))$$

where χ indicates the sheaf theoretical Euler characteristic and the H^i_{zar} are sheaf cohomology groups for the Zariski topology. $p_a(X)$ is an integer by a theorem of Serre (see [**61**, III. Theorem 5.2.(a)]).
Let X be a smooth projective k-variety. Define the geometric genus to be

$$p_g(X) := \dim_{k-\mathrm{vs}} H^0_{zar}(X,\omega_X).$$

Let X now be a projective 1-dimensional scheme over $\mathrm{Spec}(k)$ with $\tilde{X}$ its normalization. Define the geometric genus as

$$p_g(X) := \dim_{k-\mathrm{vs}} H^0_{zar}(\tilde{X},\omega_{\tilde{X}}).$$

REMARK 2.20. The two different definitions for p_g agree in the overlapping cases.

DEFINITION 2.21. Let X be a smooth projective k-variety. Define for $n \in \mathbb{N}$ the n-th pluricanonical genus (n-th plurigenus) of X to be the non-negative integer

$$P_n(X) := \dim_{k-\mathrm{vs}} H^0_{zar}(X,\omega_X^{\otimes n}).$$

The Kodaira dimension of X, denoted $\kappa(X)$, is set to $-\infty$, if all plurigenera are 0 and otherwise set to the least $m \in \mathbb{N}_0$ such that $\mathbb{N} \to \mathbb{Q}_{\geq 0}, n \mapsto P_n(X)/n^m$ is a bounded function. It holds that either $\kappa(X) = -\infty$, or $0 \leq \kappa(X) \leq \dim(X)$.

DEFINITION 2.22. Let X be a smooth projective k-surface. For $i, j \in \{0,1,2\}$ we define the Hodge numbers of X (forming the "Hodge diamond") to be the non-negative integers

$$h^{i,j}(X) := \dim_{k-\mathrm{vs}} H^j_{zar}(X, \bigwedge^i \Omega_X).$$

The irregularity of X is $q(X) := h^{0,1}(X)$.

DEFINITION 2.23. Let X be a smooth projective surface over an algebraically closed field k and let $\mathrm{Div}(X)$ be its divisor group - note that in this case Cartier divisors and Weil divisors are canonically in $1:1$-correspondence ([**61**, II.6.]). There is an intersection pairing on $\mathrm{Div}(X)$, i.e., when viewing $\mathrm{Div}(X)$ as $\mathbb{Z}$-module, there is a symmetric $\mathbb{Z}$-bilinear map $\cdot.\cdot : \mathrm{Div}(X)\times\mathrm{Div}(X) \to \mathbb{Z}$ that is determined uniquely by the following property: if $C, D \in \mathrm{Div}(X)$ are (Weil) divisors that are given by smooth curves which intersect transversally ([**61**, V.1.]), then define

$$C.D = \#(C \cap D) := \#(C \times_X D),$$

where $\#(C \times_X D)$ denotes the number of topological points of the 0-dimensional k-scheme $C \times_X D$. This pairing respects linear equivalence, algebraic equivalence, and numerical equivalence, i.e., if any of those equivalences is denoted by $\sim$ and

$C, C', D \in \mathrm{Div}(X)$, then $C \sim C' \Rightarrow C.D = C'.D$. Thus this pairing introduces a pairing on the Picard group $\mathrm{Pic}(X) \cong \mathrm{Cl}(X) := \mathrm{Div}(X)/\,\mathrm{Princ}(X)$ where $\mathrm{Princ}(X)$ are the principal divisors on X, on the Néron-Severi group $\mathrm{NS}(X) := \mathrm{Pic}(X)/\,\mathrm{Pic}^0(X)$ and on $N^1(X) := \mathrm{Div}(X)/\,\mathrm{Num}(X) \cong \mathrm{NS}(X)/\,\mathrm{NS}(X)_{tors}$ where $\mathrm{Num}(X)$ denotes the group of divisors numerically equivalent to the trivial (Weil) divisor. All three pairings are again denoted by $\cdot.\cdot$ by abuse of notation. See [**86**, I Definition 1.1.14.].
We define the Picard number $\rho := rk_{\mathbb{Z}}\,\mathrm{NS}(X)$, which is a non-negative integer (see [**38**, Corollary 9.6.17. and Remark 9.6.19.]). The lattice on $N^1(X)$ is called the Néron-Severi lattice, and by the Hodge index theorem (see [**61**, V. Remark 1.9.1.]) it is non-degenerate of signature $(1, \rho - 1)$.

Remark 2.24. For a treatment of the Picard functor Pic and a proof of $N^1(X) \cong \mathrm{NS}(X)/\,\mathrm{NS}(X)_{tors}$ for X projective over an algebraically closed field see Kleiman in [**38**, Theorem 9.6.3.]. Note that Kleiman works with invertible sheaves rather than (Weil-)divisors, but the result translates immediately for smooth X.

Proposition 2.25. *Let k be a field.*

(1) *If X, Y are birational smooth projective k-varieties, then $p_g(X) = p_g(Y)$ and $\kappa(X) = \kappa(Y)$, i.e., the geometric genus and the Kodaira dimension are birational invariants of smooth projective varieties.*
(2) *Let X, Y be birational smooth projective k-varieties, and assume either $\dim(X), \dim(Y) \leq 2$, or $\mathrm{char}(k) = 0$. Then $p_a(X) = p_a(Y)$.*
(3) *Let X be a smooth projective k-curve. Then $p_g(X) = p_a(X)$.*
(4) *Let X be a smooth projective k-surface and $\mathrm{char}(k) = 0$. Then for $i, j \in \{0, 1, 2\}$ we have $h^{i,j}(X) = h^{2-i,2-j}(X)$ by Serre duality, $h^{i,j} = h^{j,i}$ by Lefschetz principle, GAGA-theory and Poincaré duality, $h^{2,2} = h^{0,0} = 1$, because for projective k-varieties $h^{0,0} = \dim_{k-vs} H^0_{zar}(X, \bigwedge^0 \Omega_X) = \dim_{k-vs} \Gamma_X(\mathcal{O}_X) = 1$, furthermore $p_g(X) - p_a(X) = q(X) = h^{0,1}(X) = h^{1,0}(X) = h^{2,1}(X) = h^{1,2}(X)$, $P_1(X) = p_g(X) = h^{2,0}(X) = h^{0,2}(X)$, and therefore all Hodge numbers except $h^{1,1}(X)$ are birational invariants of a smooth projective surfaces.*

Proof. See [**61**, II. Theorem 8.19. and V.6.], [**61**, V. Corollary 5.6. and Remark 5.6.1.] and [**61**, III. Remark 7.12.2.]. The proofs literally translate to the non-algebraically closed case. For the last part we indicated the arguments already in the statement. □

Remark 2.26. The second part of the last proposition can be generalized. Note that by [**61**, III. Exercise 5.8.] proper 1-dimensional schemes over an algebraically closed field are projective, one can formally weaken projective to proper.
For a 1-dimensional geometrically reduced geometrically connected projective k-scheme X over an arbitrary field k one can relate the arithmetic genus to the arithmetic genera of the normalizations (which in this case serve as desingularizations by [**87**, 4. Proposition 2.24. and Remark 2.25.]) of the irreducible components of

X, their number and some invariants of the singularities – see [**61**, IV. Exercise 1.8.] in a special case or [**87**, 7. Proposition 5.4.] for the general case. Since the geometric genus is defined via the normalization, the formula linking arithmetic and geometric genus follows at once from the result above. For such non-integral "curves" embedded into surfaces these formulas can be rewritten in a simpler form (see [**61**, V. Example 3.9.2.]).

Remark 2.27. Due to the last proposition, when X is a smooth projective curve, one usually refers to the arithmetic genus and the geometric genus simply as the genus. To prevent confusion we avoid the short term.

2.1.3. Classification of Curves and Their Arithmetic

Next we discuss varieties over algebraically closed fields and their classification. Let $k = k^{alg}$ be an algebraically closed field. We make some remarks on varieties over not necessarily algebraically closed fields denoted by l for clarity. We aim at surfaces, but we start with a recount of the classification of lower dimensional varieties in a colloquial style.

The classification up to isomorphism of k-schemes of 0-dimensional k-varieties is trivial, since the only one up to isomorphism is $\mathrm{Spec}(k)$.

The classification of k-curves up to k-birational morphism is also fairly well understood. Every curve is birational to a smooth projective curve unique up to isomorphism (see [**61**, I. Corollary 6.11.] – the uniqueness is implicit there). In particular two smooth projective curves are birational if and only if they are already isomorphic.

Now the isomorphism classes of smooth projective k-curves can be partitioned by their geometric genus. One then formulates the associated moduli problem for each geometric genus $p_g \in \mathbb{N}_0$ by a functor:

$$F : \mathbf{Sch}_{/\,\mathrm{Spec}(k)} \to \mathbf{Set}, X \mapsto \mathrm{Iso}_{p_g}(X),$$

where $\mathrm{Iso}_{p_g}(X)$ is the set of isomorphism classes of flat X-schemes Y such that the (geometric) fibers are smooth curves of geometric genus p_g and Y is projective over X. This functor is not representable (in the category $\mathbf{Sch}_{/\,\mathrm{Spec}(k)}$), i.e., there is no fine moduli space of smooth projective curves of geometric genus p_g. There are coarse moduli spaces M_{p_g}, which in the case of $p_g = 0$ is just $\mathrm{Spec}(k)$, for $p_g = 1$ it is the affine line $\mathbb{A}^1_k$ called j-line in this context, and in the other cases they are quasi-projective varieties of dimension $3p_g - 3$. There exist fine moduli spaces for the analogous functors in the category of stacks. See [**60**][2].

Although this seems to be a final word on classifying curves, there are still open questions. They involve additional structure such as marked points ([**60**]) or embeddings into certain spaces like projective 3-space $\mathbb{P}^3_k$ where the degree of the embedding becomes a quantity of interest ([**61**, IV.6.]).

[2]Be aware of the colloquial style of this book; e.g., when they speak of "genus 1-curves" they usually mean "elliptic curves", i.e., demanding additionally the existence of a section of the structure morphism etc.

As sketched in section 1.1, the geometry of curves determines arithmetic. We give a proper statement of this fact:

THEOREM 2.28. *Let l be a number field and X a smooth projective l-curve whose set of l-points is non-empty, i.e., $P \in X(l) \neq \emptyset$.*

(1) *If $p_g(X) = 0$, then $X \cong \mathbb{P}^1_l$, and therefore $X(l)$ is parametrized by the naive projective l-line.*
(2) *If $p_g(X) = 1$, then X is an abelian variety (an elliptic curve in this case) and $X(l)$ carries a natural group structure via the map $X(l) \to \mathrm{Pic}^0(X), Q \mapsto \mathcal{L}_{\mathrm{im}(Q)} \otimes \mathcal{L}^{-1}_{\mathrm{im}(P)}$ (for $\mathcal{L}_{\mathrm{im}(Q)}$ see* [**61**, IV. Remark 4.10.4. and IV.1.]*) which is a finitely generated abelian group.*
(3) *If $p_g(X) \geq 2$, then $X(l)$ is a finite set.*

PROOF. The first part is in principle known for centuries, the second is part of the Mordell-Weil theorem, and the last part is the Faltings theorem (Mordell conjecture) – see [**67**] for further references. □

REMARK 2.29. Note that for smooth curves over a field l we have $p_g(X) = p_g(X^{alg})$ so one can truly say that the geometric genus as a geometric invariant (in the spirit of an invariant after going to the algebraic closure) determines the arithmetic behavior (of rational points).
The three cases are also distinguished by the behavior of the canonical line bundle ω_X and the Kodaira dimension κ_X as follows:

$$\begin{aligned} p_g = 0 &\Leftrightarrow \omega_X^{-1} \text{ is ample} &&\Leftrightarrow \kappa_X = -\infty, \\ p_g = 1 &\Leftrightarrow \omega_X \cong \mathcal{O}_X &&\Leftrightarrow \kappa_X = 0, \\ p_g \geq 2 &\Leftrightarrow \omega_X \text{ is ample} &&\Leftrightarrow \kappa_X = 1. \end{aligned}$$

REMARK 2.30. We saw that the classification over the algebraic closure was enough to qualitatively predict the arithmetic of curves. The classification (up to isomorphism) of smooth curves over arbitrary fields is interesting on its own, but much more complicated.
Milne gives an example in [**93**, IV.7.] where he classifies genus 1 curves X over $\mathbb{Q}$. Instead of the single j-invariant, i.e., a point on the j-line, the coarse moduli space for genus 1-curves over an algebraically closed field, one now needs 3 invariants: the j-invariant of the Jacobian $(E, O) := \mathrm{Jac}(X)$ of X determines the isomorphism class of X^{alg}, the associated cohomology class in $H^1_{gal}(\mathbb{Q}, \mathrm{Aut}(E, O))$ determines the elliptic curve, i.e. (E, O), of which X is a principal homogeneous space (twist of an elliptic curve), and the associated cohomology class in $H^1_{gal}(\mathbb{Q}, E)$ determines which homogeneous space X is with respect to (E, O) (twist of a homogeneous space).
In this example the classification is complete, but for geometric genus other than 1 a curve is not a homogeneous space, so this example does not generalize right on the nose.

The last remarks serve two purposes: first to give a motivation why classification of varieties is interesting from an arithmetic point of view, and second to give a glimpse at the difficulties arising in classification over non-algebraically closed fields like a number field l, which would seem to be the first idea to pursue when one is interested in l-rational points.
Since the arithmetic of surfaces is by far not as well understood as the arithmetic of curves, the theorem 2.28 serves as a model of what we would want to have for surfaces. There are results that make it fairly clear that the classification of surfaces plays an important role for their arithmetic, however not as definitive as in the curve case. See subsection 2.3.2 for further remarks in this direction.
One also can imagine that if the classification of curves over non-algebraically closed fields is such a hard problem, then the analogous problem for surfaces is even harder.

2.1.4. Classification of Surfaces

Now we finally turn to the classification of proper k-surfaces up to birational equivalence. The same conventions regarding $k = k^{alg}$ and l as in the last subsection apply.

PROPOSITION 2.31. *Any smooth proper k-surface is projective.*

PROOF. See [**62**, II.4.2]. □

PROPOSITION 2.32 (Resolution of Singularities). *Let X be either*

(1) *a k-variety of dimension ≤ 2, or*
(2) *a quasi-projective k-variety and* $\mathrm{char}(k) = 0$.

Then there is a smooth k-variety Y admitting a projective birational morphism $f : Y \to X$.
In particular, any proper k-surface is birational to a smooth projective k-surface.

PROOF. See [**76**, 3.5] (note that any surface in our sense can be normalized by [**61**, II. Exercise 3.8.] and that in Kollár's terminology being a surface does not imply reduced) and [**76**, Theorem 3.27]. We indicate the techniques involved: for 0-dimensional varieties there is nothing to do due to reducedness, for curves one can take the normalization, and for surfaces one iterates normalization and blowups in codimension 2 subschemes. For characteristic 0, one uses Hironaka's embedded resolution via blowups in smooth loci.
The second statement then follows immediately from the last proposition. □

DEFINITION 2.33. A smooth k-surface X is called a relative minimal model (of its function field) or simply relatively minimal, if and only if any birational morphism $f : X \to Y$ to another smooth surface is an isomorphism.
Let X be a smooth k-surface. A (-1)-curve C is a closed subvariety $C \subset X$ such that $C \cong \mathbb{P}^1_k$ with self-intersection $C.C = -1$.

PROPOSITION 2.34. *Let X be a smooth projective k-surface.*

(1) *Any (-1)-curve $C \subset X$ can be blown down, i.e., there is a smooth projective k-surface Y and a morphism $\pi : X \to Y$ that is a blowup in a single point with exceptional curve C.*
(2) *There is a relative minimal model Z and a birational morphism $f : X \to Z$ that factors as a finite sequence of blowdowns of (-1)-curves.*
(3) *The relative minimal model Z is unique except possibly when $\kappa(X) = -\infty$.*

PROOF. See [**61**, V. Theorem 5.7.], [**61**, V. Theorem 5.8.] and [**61**, V. Remark 5.8.4.]. Note that in [**61**, V] surfaces are assumed to be smooth and projective. □

COROLLARY 2.35. *Summarizing the above, any proper k-surface X is birational to a smooth projective relatively minimal surface Y.*

We now give the classification result for smooth projective relatively minimal k-surfaces. It is called the Enriques-Kodaira classification. It was basically obtained by Enriques as part of the results of the so called "italian school of algebraic geometry" for smooth complex algebraic surfaces and consequently by the Lefschetz principle for any smooth surfaces over any algebraically closed field of characteristic 0. It was extended by Kodaira to compact complex 2-dimensional manifolds, and finalized by Mumford and Bombieri in dealing with the characteristic $p > 0$ case. The different classes of the classification are defined in the statement of the result except for the following.

DEFINITION 2.36. A smooth projective k-surface X is an elliptically fibered (elliptic) surface, if and only if there exists a proper connected morphism $f : X \to C$ to a smooth projective k-curve C, such that all fibers are curves and the general fiber is an elliptic curve, i.e., there is a Zariski-dense subscheme $C' \subset C$, such that the fibers over C' are elliptic curves. The fibers which are not elliptic curves are called singular fibers, and f is called an elliptic fibration.
A smooth projective k-surface X is called quasi-elliptically fibered if and only if the analogous statement holds with "elliptic curve" replaced by "rational non-smooth curve with exactly one node-singularity". This is only interesting for $\mathrm{char}(k) \in \{2, 3\}$.

REMARK 2.37. An elliptically fibered surface has only finitely many singular fibers, and for relatively minimal elliptically fibered surfaces the singular fibers fall into finitely many classes given by the "Kodaira classification", which is not to be confused with the Enriques-Kodaira classification of surfaces stated in the theorem below. The determination of the singular fibers of a given elliptic fibration can be achieved by the so called "Tate's algorithm".
Some authors (e.g., Schütt and Shioda in [**113**]) also require that an elliptically fibered surface has the property that no component of a fiber is a (-1)-curve. This would still not imply relatively minimal as we defined above though.
The terminology is somewhat misleading. Elliptic curves (over some base scheme)

are relative genus 1-curves that admit a section. However, for elliptic fibrations we do not require of them to have a section. We only require the general fiber to be an elliptic curve.

THEOREM 2.38 (Enriques-Kodaira classification). *Let X be smooth projective relatively minimal k-surface with $k = k^{alg}$ and* $\mathrm{char}(k) = 0$*. Then it belongs to exactly one of the following classes:*

(1) ***Rational surfaces:*** *X is birational to $\mathbb{P}^2$. Then $\kappa(X) = -\infty$, and is isomorphic to either $\mathbb{P}^2_k$ or a relatively minimal Hirzebruch surface, which constitute a countable family.*
(2) ***Ruled surfaces of genus*** > 0***:*** *X is birational to $C \times_{\mathrm{Spec}(k)} \mathbb{P}^1_k$ where C is a smooth projective k-curve of geometric genus $p_g(C) > 0$. Then $\kappa(X) = -\infty$.*
(3) ***Abelian surfaces:*** *X is an abelian variety. Then $\kappa(X) = 0$.*
(4) ***K3 surfaces:*** *X has irregularity $q(X) = p_g(X) - p_a(X) = 0$ and trivial canonical invertible sheaf $\omega_X \cong \mathcal{O}_X$. Then $\kappa(X) = 0$.*
(5) ***Enriques surfaces:*** *X has irregularity $q(X) = p_g(X) - p_a(X) = 0$ and non-trivial canonical invertible sheaf that satisfies $\omega_X^{\otimes 2} \cong \mathcal{O}_X \ncong \omega_X$. Then $\kappa(X) = 0$, and X is always a quotient of a K3 surface by a group action of $\mathbb{Z}/2\mathbb{Z}$.*
(6) ***Bi-elliptic (or hyperelliptic) surfaces:*** *X is a quotient of $E_1 \times_{\mathrm{Spec}(k)} E_2$ by a finite group action where E_1, E_2 are elliptic curves and X is not abelian. Then $\kappa(X) = 0$.*
(7) ***Minimal properly elliptic surfaces:*** *X has Kodaira dimension $\kappa(X) = 1$. Then X is an elliptically fibered surface.*
(8) ***Surfaces of general type:*** *X has Kodaira dimension $\kappa(X) = 2$.*

PROOF. See [**9**, VI. Theorem (1.1)] for a complete proof or [**61**, V.6.] for some aspects. □

REMARK 2.39. In the classification of compact complex relatively minimal 2-dimensional manifolds, of which the algebraic surfaces over $k = \mathbb{C}$ classified above are a subclass by the GAGA-principle, there are two additional classes, namely the class of **surfaces of class VII** with Kodaira dimension $\kappa(X) = -\infty$, and the class of **primary and secondary Kodaira surfaces** with Kodaira dimension $\kappa(X) = 0$. Additionally the classes (3), (4) and (7) contain manifolds that do not arise from algebraic surfaces - the algebraic dimension is the invariant which distinguish manifolds that are not algebraic. See [**9**].
In characteristic $p > 0$ the classification is the same as in the theorem except for $p \in \{2, 3\}$. For $p = 3$ there are two additional classes namely the **quasi-bi-elliptic surfaces** with Kodaira dimension $\kappa(X) = 0$ and the **minimal properly quasi-elliptic surfaces** with Kodaira dimension $\kappa(X) = 1$. For $p = 2$ a third additional class appears, namely the **non-classical Enriques surfaces** with Kodaira dimension $\kappa(X) = 0$, which splits up in singular and super-singular Enriques

surfaces. See the original papers of Mumford and Bombieri[3] [**97**], [**16**] and [**17**]. Be aware that in arbitrary positive characteristic $p > 0$ new phenomena in the characteristic 0 classes can occur like supersingular K3 surfaces. See [**113**, 12.].
A compact overview of the results can also be obtained from the Wikipedia article on Enriques-Kodaira classification [**130**].

Remark 2.40. All classes in the Enriques-Kodaira classification are well understood except for surfaces of general type, and in the manifold case except the surfaces of class VII, which have a good description pending on the global spherical shell conjecture.

2.1.5. K3 Surfaces and Elliptically Fibered Surfaces

Diagonal quartics are elliptically fibered K3 surfaces. We now take a closer look at both, elliptically fibered surfaces and K3 surfaces. We keep the convention on the fields $k = k^{alg}$ and l. For $K3$-surfaces (in the manifold sense) over $k = \mathbb{C}$ a standard reference is [**9**, VIII]. By Lefschetz principle these results also apply, when $\mathrm{char}(k) = 0$. For elliptically fibered surfaces see Miranda's book [**96**], or the survey article [**113**], which has a section devoted to elliptically fibered K3 surfaces, and summarizes results on K3 surfaces for k of arbitrary characteristic, too.

Remark 2.41. We use the invariants introduced above for algebraic varieties also for compact complex manifolds (see [**51**]) without explicitly defining them whenever the definitions parallel each other, e.g., p_g, p_a, $h^{i,j}$, etc.

Definition 2.42. We call a relatively minimal compact complex (in particular smooth) 2-dimensional manifold X a complex K3 surface, if and only if $\omega_X \cong \mathcal{O}_{X,hol}$ and $q(X) = \dim_{k-\mathrm{vs}} H^1_{hol}(X, \mathcal{O}_{X,hol}) = 0$, where all sheaves are holomorphic sheaves.
A relatively minimal smooth projective surface X is called an algebraic K3 surface if and only if $\omega_X \cong \mathcal{O}_X$ and $q(X) = \dim_{k-\mathrm{vs}} H^1_{Zar}(X, \mathcal{O}_X) = 0$ where all sheaves are sheaves for the Zariski topology.

Remark 2.43. Let V be an l-scheme for some field l. Then we say that V is defined over a subfield $l' \subset l$, if and only if there is an l'-scheme V' such that $V \cong V' \times_{\mathrm{Spec}(l')} \mathrm{Spec}(l)$. In this case l' is called a field of definition of V. Any projective l-variety V can be defined as the 0-locus of finitely many polynomial equations having only finitely many coefficients in some projective space over l and these coefficients define by adjunction to the prime field f of l a field of definition l' that is finitely generated over f.
Now assume that we have an l-variety V with $\mathrm{char}(l) = 0$. Let l' be a field of definition finitely generated over $\mathbb{Q}$ and V' an associated l'-variety. Clearly there exists embeddings of l' into $\mathbb{C}$ - choose one. Then $V' \times_{\mathrm{Spec}(l')} \mathrm{Spec}(\mathbb{C})$ gives a $\mathbb{C}$-variety, which by GAGA-theory (see [**61**, Appendix B]) can be viewed uniquely

[3]The author was informed in private communication that these articles are a little sketchy regarding some details.

as a complex manifold.
If we require V not only to be projective but projectively embedded, i.e., with fixed closed immersion into some $\mathbb{P}^n$, and the fields of definition to be compatible with this embedding, then there is also a unique minimal field of definition l'' (cf. [**91**]). In this case the only ambiguity arises from the choice of embedding of l'' into $\mathbb{C}$. Up to this choice V yields a unique smooth $\mathbb{C}$-manifold M associated to $V'' \times_{\mathrm{Spec}(l'')} \mathrm{Spec}(\mathbb{C})$.
It turns out that V is an algebraic K3 surface if and only if M is a projective complex K3 surface. Thus K3 surfaces over characteristic 0 fields can be treated using analytic methods. This also works for more general varieties. We call this kind of argument "reduction to the complex case".

Proposition 2.44. *Let X be a (complex or algebraic) K3 surface over k. Then the Hodge numbers are $h^{0,0} = h^{2,2} = 1 = h^{2,0} = h^{0,2}$, $h^{1,0} = h^{1,2} = h^{2,1} = h^{0,1} = 0$ and $h^{1,1} = 20$. We have: $\mathrm{Pic}^0(X) = \mathrm{Pic}^\tau(X) \cong \{0\}$. Thus it holds that $\mathrm{Pic}(X) \cong \mathrm{NS}(X) \cong N^1(X)$ is a finitely generated torsion free abelian group, and we can endow $\mathrm{Pic}(X)$ and $\mathrm{NS}(X)$ with the lattice structure of $N^1(X)$, that is a non-degenerate pairing. X is simply connected, i.e., X has trivial fundamental group.*

Proof. See [**108**, Proposition 1.2.3. and Lemma 1.3.2.] for algebraic K3 surfaces, [**9**, VIII. Proposition (3.4) and Proposition (3.6)] for complex K3 surfaces, and [**38**, Theorem 9.6.3.] for the equivalence of numerically trivial and algebraically trivial up to torsion. Alternatively check [**13**, Propsition 2.10. and 2.11.] in the complex case.
By [**9**, VIII.2.] complex K3 surfaces have $\pi_1^{an}(X) = 0$. By 3.48 and reduction to the complex case it follows that $\pi_1^{ét}(X) = 0$ for algebraic K3 surfaces over a field with $\mathrm{char}(k) = 0$. For $\mathrm{char}(k) = p > 0$ we find a deformation to a characteristic 0 case by [**32**, Exposeé IV. Corollaire 1.8.] and then apply surjectivity of the induced morphism of the ètale fundamental group of the open fiber to the closed fiber from [**52**, Exposeé X, Corollaire 2.4.]. □

Proposition 2.45. *Let $f : X \to C$ be an elliptic fibration of a k-surface X. Then $\mathrm{Pic}^0(X) = \mathrm{Pic}^\tau(X)$, $\mathrm{NS}(X) \cong N^1(X)$, and we can endow $\mathrm{NS}(X)$ with the lattice structure of $N^1(X)$. Furthermore $\mathrm{Pic}^0(C) \xrightarrow{f^* \cong} \mathrm{Pic}^0(X)$ is an isomorphism, and the Picard number satisfies Igusa's inequality $\rho(X) \leq h^{2,0}+h^{1,1}+h^{0,2}$. If $\mathrm{char}(k) = 0$ and no fibers of f are (-1)-curves, then $h^{0,0} = h^{2,2} = 1$, $h^{2,0} = h^{0,2} = p_g(X)$, $h^{1,0} = h^{1,2} = h^{2,1} = h^{0,1} = q(X) = p_g(C)$, $h^{1,1} = 10p_a(X) + 2q(X) + 10$, and the Picard number satisfies Lefschetz's bound $\rho(X) \leq h^{1,1}$.*

Proof. See [**113**, Theorem 6.2., 6.5. and 6.12.] and [**113**, section 6.10.] for the results in arbitrary characteristic, and [**113**, sections 6.9. and 6.10] for the characteristic 0 case after reducing to the complex case. □

For the moduli space of K3 surfaces, we concentrate on complex K3 surfaces, and assume $k = \mathbb{C}$. For moduli spaces of K3 surfaces over general fields, see [**108**].

Remark 2.46. For a compact complex manifold X, we have the exponential sheaf sequence (see [**61**, Appendix B 5.])

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

and its associated long exact sequence in cohomology. Using this, one can relate the Picard group $\mathrm{Pic}(X) \cong H^1_{hol}(X, \mathcal{O}_X^*)$ to $H^2_{hol}(X, \mathbb{Z})$. We have for complex K3 surfaces:

$$H^2_{hol}(X, \mathbb{Z}) \cong \Lambda := \mathbb{H}^{\oplus 3} \oplus (-E_8)^{\oplus 2},$$

where Λ is called K3 lattice, $\mathbb{H}$ is the lattice of the hyperbolic plane and $(-E_8)$ is the negative E_8-lattice (see [**113**] or [**9**] for quick introductions to lattice theory and further references). Note that Λ is an even lattice.
A compact complex manifold X arises from a smooth proper $\mathbb{C}$-variety, if X is a submanifold of some projective space $\mathbb{P}^n$ by a theorem of Chow (see [**61**, Appendix B Theorem 2.2.]). Since all smooth proper $\mathbb{C}$-surfaces are projective by proposition 2.31, this implication is strengthened to an equivalence for surfaces. Thus a complex K3 surface arises from an algebraic K3 surface, exactly if it is projective. This is the case, if and only if it admits an ample invertible sheaf.

Definition 2.47. Let X be a complex K3 surface. A marking is a choice of an isomorphism $\phi : H^2_{hol}(X, \mathbb{Z}) \to \Lambda$ and (X, ϕ) is called a marked K3 surface.
Let X be a complex or algebraic K3 surface. A polarization is a choice of an (equivalence class of) invertible sheaf $\mathcal{L}$ on X such that it is ample. The selfintersection number $\mathcal{L}.\mathcal{L} =: 2d \in 2\mathbb{N}$ is an even integer since Λ is an even lattice and is called the degree of the polarization. Such a K3 surface is called polarizable (in degree $2d$), and the pair $(X, \mathcal{L})$ is called a polarized K3 surface. $(X, \mathcal{L})$ is called primitively polarized, if and only if $\mathcal{L}$ is primitive, if and only if there is no invertible sheaf $\mathcal{M}$ and $n \in \mathbb{N}_{>1}$ with $\mathcal{M}^{\otimes n} \cong \mathcal{L}$.
One may think of a polarization $\mathcal{L}$ as giving an embedding of lattices $\mathbb{Z} \hookrightarrow \mathrm{NS}(X)$ where 1 is mapped to (the class of) $\mathcal{L}$ and the lattice structure on $\mathbb{Z}$ is fixed by demanding compatibility. Similarly for every sublattice $M \subset \Lambda$ of signature $(1, r)$ (necessarily even and non-degenerate) a choice of an embedding of lattices $M \hookrightarrow \mathrm{NS}(X)$ is called an M-polarization (lattice polarization by M), and if it is embedded as a primitive sublattice of $\mathrm{NS}(X)$, it is called a primitive M-polarization.

Remark 2.48. Polarizable complex K3 surfaces are exactly the projective complex K3 surfaces. Such K3 surfaces may be polarizeable in different degrees.

Proposition 2.49. *There exists a fine moduli space of marked complex K3 surfaces which is a* 20*-dimensional non-Hausdorff complex manifold.*
For each $2d \in 2\mathbb{N}$ *there is a* 19*-dimensional coarse moduli space of primitively polarized K3 surfaces of degree* $2d$*.*

Let M be a suitable lattice as above of signature $(1, r)$. Then the M-polarized K3 surfaces admit local moduli spaces of dimension $19 - r$.

PROOF. See [**9**, VIII. Theorem (12.1)], the part immediately after [**108**, Corollary 1.4.14.] and [**113**, Proposition 12.6]. □

REMARK 2.50. Important notions related to the classification of complex K3 surfaces are the Torelli theorem, which comes in different variants, the period domain and period map, as well as hermitian symmetric spaces. Compare the given references.

EXAMPLE 2.51. In low degrees there is a complete picture of polarizable K3 surfaces. The following classes make up all K3 surfaces up to isomorphism of the given degree $2d \in 2\mathbb{N}$:

(1) $2d = 2$: all surfaces X which admit a degree 2 finite morphism $f : X \to \mathbb{P}^2$ with branch locus a plane smooth degree 6 curve $C \subset \mathbb{P}^2$, i.e., double covers of the plane branched in smooth sextics,
(2) $2d = 4$: all degree 4 codimension 1 smooth subvarieties $X \subset \mathbb{P}^3$, i.e., quartic surfaces,
(3) $2d = 6$: all smooth complete intersections of a degree 2 smooth hypersurface $X_1 \subset \mathbb{P}^4$ with a degree 3 smooth hypersurface $X_2 \subset \mathbb{P}^4$, i.e., complete intersections of degree 2 and 3 hypersurfaces,
(4) $2d = 8$: all smooth complete intersections of three degree 2 smooth hypersurface $X_1, X_2, X_3 \subset \mathbb{P}^5$.

When one takes an abelian surface A and forms the quotient by the $\mathbb{Z}/2\mathbb{Z}$-action given by $a \mapsto -a$, one gets a surface with 16 point singularities. Resolving these singularities gives a K3 surface which is embeddable as a degree 4 surface into $\mathbb{P}^3$. Such a surface is called a Kummer surfaces.

To summarize the results on moduli spaces: diagonal quartics over number fields, which we study in examples later on, constitute (after base change to $\mathbb{C}$) a single point in the 19-dimensional moduli space of quartic K3 surfaces, which in turn are one of the countably many codimension 1 families representing all projective complex K3 surfaces in the 20-dimensional moduli space of all complex K3 surfaces.

We give now some results on K3 surfaces which admit elliptic fibrations. Now $k = k^{alg}$ can be an arbitrary algebraically closed field.

PROPOSITION 2.52. *A (complex or algebraic) K3 surface X admits an elliptic fibration $f : X \to C$ (or a quasi-elliptic fibration), if and only if there is some divisor class $0 \neq D \in \mathrm{NS}(X)$ with $D.D = 0$.*
Let X be a (complex or algebraic) K3 surface with Picard number ρ. Then

(1) *$\rho \geq 5 \Rightarrow X$ admits an elliptic fibration,*
(2) *$\rho \in \{2, 3, 4\} \Rightarrow X$ might or might not admit an elliptic fibration,*
(3) *$\rho \leq 1 \Rightarrow X$ admits no elliptic fibration.*

If a (complex or algebraic) K3 surface X admits an elliptic fibration $f : X \to C$, then $C \cong \mathbb{P}^1$.

Proof. See [**113**, Proposition 12.10] and the following remarks. For the last statement combine propositions 2.44 and 2.45 to get $0 \cong \mathrm{Pic}^0(X) \cong \mathrm{Pic}^0(C)$. Since, up to isomorphism, $\mathbb{P}^1$ is the only smooth curve with trivial Jacobian, and since over algebraically closed fields k we have $\mathrm{Jac}(C)(k) \cong \mathrm{Pic}^0(C)$, we are done. □

Remark 2.53. Combining the results on moduli of lattice polarized K3 surfaces with the last proposition and taking into account that Λ clearly has an appropriate signature $(1, 4)$-sublattice we see that there are $19-4 = 15$-dimensional subspaces of the moduli space of K3 surfaces that parametrize K3 surfaces admitting elliptic fibrations. In view of the 20-dimensional moduli space of complex K3 surfaces, elliptic fibrations should be considered a special but not really exotic phenomenon.

Below we discuss elliptically fibered surfaces again, but since the application require the possibility of non-algebraically closed base fields, we do not treat those arithmetic notions here in the geometric section.

2.2. Adeles, Hasse Principle and Local Solubility

This section is devoted to arithmetic. For standard references to algebraic number theory see the books of Cassels and Fröhlich [**20**], Fröhlich and Taylor [**42**], or Neukirch [**100**]. We merely give a summery on adeles and adelic points on varieties and the related notions of Hasse principle and weak approximation and outline a well-known method to decide local solubility, of which we know no pleasing reference for general varieties.

2.2.1. Adeles

The following definitions encompass many nontrivial results. Please check the references above for details.

Definition 2.54. Let k be a field. A valuation is a non-constant map $|\cdot| : k \to \mathbb{R}_{\geq 0}, a \mapsto |a|$ satisfying

(1) $\forall a \in k : |a| = 0 \Leftrightarrow a = 0_k$,
(2) $\forall a, b \in k : |ab| = |a||b|$,
(3) $\forall a, b \in k : |a + b| \leq |a| + |b|$.

Two valuations $|\cdot|_1$, $|\cdot|_2$ are called equivalent, if and only if there is a $r \in \mathbb{R}_{>0}$ with $|\cdot|_1 = |\cdot|_2^r$. A valuation $|\cdot|$ is called discrete, if $|k \setminus \{0\}| \subset \mathbb{R}_{>0}$ is discrete. It is called non-archimedian, if additionally $\forall a, b \in k : |a + b| \leq \max\{|a|, |b|\}$, and archimedian otherwise. Being discrete or archimedian depends only on the equivalence class.
The set of equivalence classes of valuations is called the set of places $\Omega_k = \Omega_k^\infty \dot\cup \Omega_k^0$

partitioned in archimedian (infinite) and non-archimedian (finite) places. For a place $\mathfrak{p} \in \Omega_k$ we denote a representing valuation by $|\cdot|_\mathfrak{p}$.
For a non-archimedian place $\mathfrak{p}$ we define the associated valuation ring $\mathfrak{o}_{k,\mathfrak{p}} := \mathfrak{o}_\mathfrak{p} := \{a \in k : |a|_\mathfrak{p} \leq 1\}$, the associated valuation ideal by $\mathfrak{p}_k := \mathfrak{p} := \{a \in k : |a|_\mathfrak{p} < 1\}$, and the associated residue field by $\kappa_{k,\mathfrak{p}} := \kappa_\mathfrak{p} := \mathfrak{o}_\mathfrak{p}/\mathfrak{p}$.
By $k_\mathfrak{p}$ we denote the completion of k with respect to $|\cdot|_\mathfrak{p}$. It is a complete topological space and carries a valuation, which is the unique extension of $|\cdot|_\mathfrak{p}$, denoted the same.

Remark 2.55. Finite places are also called primes. There is a close relationship between prime ideals of height 1 and primes of valuations, provided that the concerned ring is nice enough (see, e.g., [**61**, II.6.] for a geometric variant).

Definition 2.56. Let k be a field. It is a global field if and only if

(1) there is a set of representatives $|\cdot|_\mathfrak{p}$ of Ω_k such that

$$\forall a \in k^* := k \setminus \{0_k\} : \#\{\mathfrak{p} \in \Omega_k : |a|_\mathfrak{p} \neq 1\} < \infty \text{ and}$$
$$\forall a \in k^* : \prod_{\mathfrak{p}\in\Omega_k} |a|_\mathfrak{p} = \prod_{\mathfrak{p}\in\{\mathfrak{p}\in\Omega_k : |a|_\mathfrak{p}\neq 1\}} |a|_\mathfrak{p} = 1,$$

(2) and $\Omega_k^\infty \neq \emptyset$, or $\exists \mathfrak{p} \in \Omega_k^0 : \#\kappa_\mathfrak{p} < \infty$.

A global field is either a number field, i.e., a finite field extension of $\mathbb{Q}$ or a function field (of a projective curve over a finite field), i.e., a finite field extension of $\mathbb{F}_p(t)$ for some prime number p. Note that necessarily $\#\Omega_k^\infty < \infty$ and that all valuations $|\cdot|$ representing a place in Ω_k^0 are discrete.
There is a subring of k, which we denote by $\mathfrak{o}_k$, unique up to isomorphism of rings, which is a Dedekind domain with quotient field k and with all non-zero prime ideals of finite index. For a number field k it is the associated number ring, i.e., the integral closure of $\mathbb{Z}$, and for a function field k extending $\mathbb{F}_p(t)$ it is the integral closure of $\mathbb{F}_p[t]$ in k.
A local field is the completion of a global field. It is either a finite field extension of the p-adic numbers $\mathbb{Q}_p$ or of the formal power series $\mathbb{F}_p((t))$ in the non-archimedian case (non-archimedian local field), or $\mathbb{R}$ or $\mathbb{C}$ otherwise (archimedian local field). If $\mathfrak{p} \in \Omega_k^\infty$, then $k_\mathfrak{p}$ is either $\mathbb{R}$, or $\mathbb{C}$. This induces a partition $\Omega_k^\infty = \Omega_k^\mathbb{R} \dot\cup \Omega_k^\mathbb{C}$.

Remark 2.57. There is the notion of normalized valuation for a global field. We assume that our valuation representing places of a global field are normalized. See, e.g., [**20**, II.7. and II.11.].
To any valuation as defined above there is an associated (logarithmic) valuation, which we denote $v_\mathfrak{p} : k \to \mathbb{R}\dot\cup\{\infty\}$. If $\mathfrak{p}$ is non-archimedian, we assume that $v_\mathfrak{p}(k^*) = \mathbb{Z}$, i.e., it is a normalized discrete (logarithmic) valuation.

Remark 2.58. The notions of local field and local ring are totally different and have almost nothing to do with each other, but are easily confused. For a local field k the ring $\mathfrak{o}_k$ is a complete local ring.

Remark 2.59. All finite extensions of global (local) fields are necessarily again global (local) fields and are separable.

Remark 2.60. If we weaken the non-archimedian part of the second condition for global fields to $\mathrm{char}(\kappa_\mathfrak{p}) < \infty$, more general function fields would satisfy the axioms. Some of the theory is still valid for them, see [**5**], [**39**], or for a recent application [**106**], but, e.g., the last remark is no longer valid in general, which is why separability conditions such as perfect residue fields are often necessary.

Remark 2.61. We use the term "almost all" for "all except possibly finitely many".

Definition 2.62. Let I be an index set, $I^\infty \subset I$ a finite subset and $I^0 := I \setminus I^\infty$. Let $(T_i)_{i\in I}$ be a family of topological spaces and $(U_i)_{i\in I^0}$ a family of open subsets $\forall i \in I^0 : U_i \subset T_i$. We define the restricted product for this data to be the set

$$\prod_{i\in I} T_i := \bigcup_{J\subset I^0, \# J<\infty} \left(\prod_{i\in I^\infty \dot\cup J} T_i \times \prod_{i\in I^0\setminus J} U_i \right),$$

together with the topology induced by the following basis:

$$\mathcal{B} := \{V \subset \prod_{i\in I} T_i : \exists J \subset I^0 : \# J < \infty \wedge \forall i \in I^\infty \dot\cup J : \exists V_i \subset^{open} T_i : V = \prod_{i\in I^\infty \dot\cup J} V_i \times \prod_{i\in I^0\setminus J} U_i\}.$$

Moreover $\prod_{i\in I} T_i$ is a locally compact topological space, if all T_i are locally compact and almost all U_i are compact.

Remark 2.63. The restricted product depends on the choice of the U_i, but choosing only finitely many U_i different does not affect it. This choice is not reflected by the notation, but will be clear from the context.
If the (T_i, Σ_i, μ_i) are measure spaces with $\forall i \in I^0 : \mu_i(U_i) = 1$, then there is a restricted product measure space on $\prod_{i\in I} T_i$. This is important for the definition of Tamagawa numbers, which play a prominent role in some quantitative conjectures on the asymptotics of rational points below given height. See, e.g., the Habilitationsschrift of Jahnel [**73**] for a detailed introduction.

Definition 2.64. Let k be a global ring. Set $I := \Omega_k, I^\infty := \Omega_k^\infty, \forall \mathfrak{p} \in \Omega_k : T_\mathfrak{p} := k_\mathfrak{p}$ and $\forall \mathfrak{p} \in \Omega_k^0 : U_\mathfrak{p} := \mathfrak{o}_\mathfrak{p}$. Then the k-adeles are the restricted product

$$\mathbb{A}_k := \prod_{\mathfrak{p}\in\Omega_k} k_\mathfrak{p},$$

which furnished with pointwise addition and multiplication is a locally compact topological ring due to the (local) compactness of $\mathfrak{o}_\mathfrak{p}$ and $k_\mathfrak{p}$.
The k-ideles are

$$\mathbb{I}_k := \{(x, y) \in \mathbb{A}_k \times \mathbb{A}_k : xy = 1_{\mathbb{A}_k}\},$$

which form a topological group under multiplication with subspace topology of $\mathbb{A}_k \times \mathbb{A}_k$ called the idele topology. The ideles are naturally isomorphic to the multiplicative group of the adele ring $\mathbb{A}_k$ and are considered a subset of the adeles.

By general properties of (co-)products and (co-)limits we have for any pair $\mathfrak{p}', S'$ with $\Omega_k^\infty \cup \{\mathfrak{p}'\} \subset S' \subset \Omega_k$ and $|S'| < \infty$, that

$$k_{\mathfrak{p}'} \hookrightarrow \oplus_{\mathfrak{p}\in S'} k_{\mathfrak{p}} \cong \prod_{\mathfrak{p}\in S'} k_{\mathfrak{p}} \hookrightarrow \prod_{\mathfrak{p}\in S'} k_{\mathfrak{p}} \oplus \prod_{\mathfrak{p}\in\Omega\setminus S'} \mathfrak{o}_{k_{\mathfrak{p}}} \cong \prod_{\mathfrak{p}\in S'} k_{\mathfrak{p}} \times \prod_{\mathfrak{p}\in\Omega\setminus S'} \mathfrak{o}_{k_{\mathfrak{p}}} =: \mathbb{A}_{S'}$$

and thus an embedding $k_{\mathfrak{p}'} \hookrightarrow \varinjlim_{\Omega_k^\infty \cup\{\mathfrak{p}'\}\subset S\subset\Omega_k, |S|<\infty} \mathbb{A}_S \cong \varinjlim_{\Omega_k^\infty\subset S\subset\Omega_k, |S|<\infty} \mathbb{A}_S = \mathbb{A}_k$. On the other hand we have projections for any finite S' with $\mathfrak{p}' \in S'$

$$\mathbb{A}_{S'} = \prod_{\mathfrak{p}\in S'} k_{\mathfrak{p}} \times \prod_{\mathfrak{p}\in\Omega\setminus S'} \mathfrak{o}_{k_{\mathfrak{p}}} \xrightarrow{\pi_{S'}} k_{\mathfrak{p}'}$$

forming an inductive system and thus giving rise to $\mathbb{A}_k = \varinjlim_{\Omega_k^\infty\subset S\subset\Omega_k, |S|<\infty} \mathbb{A}_S \cong \varinjlim_{\Omega_k^\infty\cup\{\mathfrak{p}'\}\subset S\subset\Omega_k, |S|<\infty} \mathbb{A}_S \twoheadrightarrow k_{\mathfrak{p}'}$.

In summary we get two morphisms whose composition $k_{\mathfrak{p}'} \hookrightarrow \mathbb{A}_k \twoheadrightarrow k_{\mathfrak{p}'}$ is the identity $\mathrm{id}_{k_{\mathfrak{p}'}}$ as is rapidly seen by inspection.

There are natural inclusions $\iota_{\mathbb{A}_k} : k \hookrightarrow \mathbb{A}_k$ and $k^* \hookrightarrow \mathbb{I}_k$ induced as the diagonal morphism of the various embeddings $\iota_{k_{\mathfrak{p}'}} : k \hookrightarrow k_{\mathfrak{p}'}$ for $\mathfrak{p}' \in \Omega_k$.

Remark 2.65. The k-adeles are compatible with finite field extensions K/k and are a non-Noetherian ring. The adelic topology on $\mathbb{I}_k$, which is the subspace topology with respect to $\mathbb{A}_k$, is not the same as the idelic topology on $\mathbb{I}_k$. $\mathbb{I}_k$ is not a topological group with respect to the adelic topology.

Remark 2.66. One can also introduce adeles for generalized global fields. Many results carry over quite well, often just requiring some additional separability conditions as already mentioned above.

Remark 2.67. With the help of the ideles for a global field k one can define the idele class group $\mathrm{Cl}_{\mathbb{I}}(\mathfrak{o}_k)$, a slightly finer invariant than the ideal class group $\mathrm{Cl}(\mathfrak{o}_k) \cong \mathrm{Pic}(\mathrm{Spec}(\mathfrak{o}_k))$. The additional information comes from the infinite places of k which can not be encoded in the ideals of the associated ring $\mathfrak{o}_k$. This is the basic idea of Arakelov geometry, which tries to add equivalent data from archimedian places to usual objects, e.g., locally free sheaves get enriched by real or complex vector bundles, etc.

We summarize the results in an exemplary diagram for $k = \mathbb{Q}$, which also contains localization and Henselization at some non-zero primes (p) for completness. One can interpret the different levels of the diagram geometrically: the first is the space itself, the second a basis for the Zariski topology, the third represents étale topology (which is not a topology in the usual sense, but a Grothendieck topology), the next a kind of vague analytic topology made up by local fields (and associated rings), and the last contains the points of the space itself and for reasons of presentation the restricted product of the local fields. We identify a non-archimedian place with the associated prime ideal $\mathfrak{p} = (p)$ and only display the behavior at two finite places.

$$\begin{tikzcd}
& \mathbb{Z} = \mathfrak{o}_{\mathbb{Q}} \arrow[dl, hook] \arrow[d, hook] \arrow[dddr, hook] \arrow[drrr, hook] & & & & \\
\mathbb{Z}_{(p_1)} \arrow[d, hook] & \mathbb{Z}_{(p_2)} \arrow[d, hook] & & & \mathbb{Q} \arrow[ddl, hook] \arrow[dd, hook] \arrow[ddr, hook] \arrow[ddd, hook, bend right] & \\
\mathbb{Z}^h_{(p_1)} \arrow[d, hook] & \mathbb{Z}^h_{(p_2)} \arrow[d, hook] & & & & \\
\hat{\mathbb{Z}}^{(p_1)} =: \mathbb{Z}_{p_1} \arrow[d] \arrow[rrr, hook, bend right] & \hat{\mathbb{Z}}^{(p_2)} =: \mathbb{Z}_{p_2} \arrow[d] \arrow[rrr, hook, bend right] & \mathbb{R} \arrow[rrr, hook, bend right] & \hat{\mathbb{Q}}^{(p_1)} =: \mathbb{Q}_{p_1} \arrow[dr, hook] & \hat{\mathbb{Q}}^{(p_2)} =: \mathbb{Q}_{p_2} \arrow[d, hook] & \mathbb{R} \arrow[dl, hook] \\
\mathbb{F}_{p_1} = \kappa_{(p_1)} & \mathbb{F}_{p_2} = \kappa_{(p_2)} & & & \mathbb{A}_{\mathbb{Q}} \arrow[r, hook] & \prod_{\mathfrak{p} \in \Omega_{\mathbb{Q}}} \mathbb{Q}_{\mathfrak{p}}
\end{tikzcd}$$

2.2.2. The Hasse Principle and Weak Approximation

Until the end of this chapter let k be a number field. Most of the following should hold more generally for global fields. Let X be a k-variety.

Remark 2.68. Since $k_\mathfrak{p}, \mathbb{A}_k$ and $\prod_{\mathfrak{p}\in\Omega_k} k_\mathfrak{p}$ are topological rings R, the associated sets of R-valued points $X(R)$ resp. $\prod_i X(R_i)$ carry a (product) topology, and can thus be considered topological spaces instead of mere sets (see [**104**, 2.3.9.]).

Proposition 2.69. *The morphisms of the above diagram of rings R give rise to natural morphisms for the corresponding R-rational points. Explicitly:*

$$X(k) \xrightarrow{\iota_X} X(\mathbb{A}_k) \xrightarrow{\phi_X} X(\prod_{\mathfrak{p}\in\Omega_k} k_\mathfrak{p}) \xrightarrow{\psi_X} \prod_{\mathfrak{p}\in\Omega_k} X(k_\mathfrak{p}).$$

ι_X is an embedding, ϕ_X is a continuous embedding, and ψ_X is an topological isomorphism. If X is additionally proper, e.g., projective, then ϕ_X is also a topological isomorphism.

Proof. The statements about maps behaving well with respect to topology are immediate from the definition of adelic topology and product topology. For the statements about ϕ_X and ψ_X see [**73**, III. Lemmas 1.9. and 3.2.], taking into account the properties of k-varieties.
Would ι_X not be injective, then there would be two different points $x_1, x_2 : \operatorname{Spec}(k) \to X$ in $X(k)$ defining the same point $x_\mathfrak{p}$ in $X(k_\mathfrak{p})$ for all $\mathfrak{p}$. The maps x_1, x_2 on the underlying topological spaces must agree, so we may restrict to an affine open chart isomorphic to $\operatorname{Spec}(S)$ of X containing the image of x_1, where S is a finite type k-algebra. Thus we would get two different ring morphisms $\alpha_1, \alpha_2 : S \to k$ which agree after composing with the injective $\iota_\mathfrak{p}$, and we get a contradiction. $\square$

REMARK 2.70. Analogous statements are more generally true for varieties over a global field k or arithmetic varieties over $\mathfrak{o}_k$.

DEFINITION 2.71. Let $\mathbf{X}$ be a set of k-varieties, which is referred to as a class of k-varieties as is customary. Let $S \subset^{finite} \Omega_k$. We say that $\mathbf{X}$ satisfies

(1) the Hasse principle (local-to-global principle, HP)
$$:\Leftrightarrow (\forall X \in \mathbf{X} : X(k) \neq \emptyset \Leftrightarrow X(\prod_{\mathfrak{p}\in\Omega_k} k_\mathfrak{p}) \neq \emptyset),$$
(2) weak approximation away from S (WA_S)
$$:\Leftrightarrow (\forall X \in \mathbf{X} : \overline{\mu(X(k))}^{prod} = X(\prod_{\mathfrak{p}\in\Omega_k\setminus S} k_\mathfrak{p}),$$
where the closure is taken with respect to product topology and μ is the canonical map (not necessarily an inclusion),
(3) weak approximation (WA) $:\Leftrightarrow$ $\mathbf{X}$ satisfies $\mathrm{WA}_\emptyset$,
(4) strong approximation away from S (SA_S)
$$:\Leftrightarrow (\forall X \in \mathbf{X} : \overline{\nu(X(k))}^{\mathbb{A}} = X(\mathbb{A}_k/\prod_{\mathfrak{p}\in S} k_\mathfrak{p}),$$
where the closure is taken with respect to the quotient topology induced by the adelic topology and ν is the canonical map (not necessarily an inclusion),
(5) strong approximation (SA) $:\Leftrightarrow$ $\mathbf{X}$ satisfies $\mathrm{SA}_\emptyset$.

We say that HP holds for a k-variety X, if and only if HP holds for $\{X\}$; analogously for the other notions just defined.

REMARK 2.72. Let $S \subset^{finite} \Omega_k$. If a class $\mathbf{X}$ satisfies SA_S, then it satisfies WA_S.
If $\mathbf{X}$ satisfies WA, it satisfies HP.
For $S \subset S'$ we clearly have WA_S implies $\mathrm{WA}_{S'}$ and SA_S implies $\mathrm{SA}_{S'}$.
An equivalent formulation of WA is as follows:
$$\forall X \in \mathbf{X} : \forall \Sigma \subset^{finite} \Omega_k : \forall \epsilon > 0 : \forall (x_\mathfrak{p})_{\mathfrak{p}\in\Sigma} \in \prod_{\mathfrak{p}\in\Sigma} X(k_\mathfrak{p}) : \exists x \in X(k) : \forall \mathfrak{p} \in \Sigma : |x_\mathfrak{p} - \iota_{X,\mathfrak{p}}(x)|_\mathfrak{p} < \epsilon.$$
Using the topological isomorphism ψ_X one sees that $\mathbf{X}$ satisfies HP, if and only if
$$\forall X \in \mathbf{X} : X(k) \neq \emptyset \Leftrightarrow \forall \mathfrak{p} \in \Omega_k : X(k_\mathfrak{p}) \neq \emptyset.$$

PROPOSITION 2.73. *Let $S \subset^{finite} \Omega_k$. If $\mathbf{X}$ contains only proper k-varieties, or even only projective k-varieties, then:*

(1) $\mathbf{X}$ *satisfies* $WA_S \Leftrightarrow \mathbf{X}$ *satisfies* SA_S.
(2) $\mathbf{X}$ *satisfies HP* $\Leftrightarrow \forall X \in \mathbf{X} : X(k) \neq \emptyset \Leftrightarrow X(\mathbb{A}_k) \neq \emptyset$.

PROOF. Apply proposition 2.69 to the definitions. $\square$

Remark 2.74. Weak approximation is a k-birational property of smooth k-varieties, i.e., if X and X' are two smooth birational k-varieties, then X satisfies WA if and only if X' does.

Example 2.75. The classical results on weak and strong approximation as in [**20**, II.6. and II.15.] reformulated in geometric terms say, that for the affine line $\mathbb{A}_k^1$ weak approximation holds and strong approximation holds away from any $S \subset \Omega_k$ with $\#S > 0$.

Example 2.76 (Hasse-Minkowski theorem). The class of smooth hypersurfaces in $\mathbb{P}^n$ of degree 2, i.e., smooth projective quadrics, satisfies WA and therefore HP.

Remark 2.77. There are many classes of varieties for which WA or HP is known to hold. The class of varieties birational over the base field to some $\mathbb{P}^n$ satisfies HP. Certain subclasses, e.g., del Pezzo surfaces of high enough degree satisfy WA. Birch proved in [**12**] using analytic methods like the circle method, that any quasi-projective subvariety $X \subset \mathbb{P}^{n-1}$ defined by r homogeneous equations of degree $\leq d$ and singular locus of dimension less than $n-1-r(r+1)(d-1)2^{(d-1)}$ satisfies HP. Thus smooth projective intersections of two quadrics in large enough dimension or cubics in large enough dimension satisfy HP. See also [**120**, Ch. 5].

2.2.3. Testing Everywhere Local Solubility

Again let k be a number field, and let X be a k-variety. This part is devoted to the so called local analysis for rational points on X.

Definition 2.78. X is called everywhere locally solvable (els), if and only if $\forall \mathfrak{p} \in \Omega_k : X(k_\mathfrak{p}) \neq \emptyset$.

Remark 2.79. In the light of the previous subsections, we have that $X(k) \neq \emptyset$ implies X is everywhere locally solvable. The inverse holds exactly, when X satisfies HP. This underlines the significance of HP and everywhere local solubility for questions on rational points. Thus it is important to have a method to test everywhere local solubility, which we outline now.

We begin with some remarks about the representation of a variety X and comment on why we restrict to smooth projective k-varieties. Then we recount several results: the Weil conjectures, arithmetic models with smooth reduction modulo $\mathfrak{p}$, Hensel's lemma and decidability of local solubility for a single or finitely many $\mathfrak{p}$, and behavior at archimedian places. Finally we combine these results for the method to decide everywhere local solubility and discuss some examples most notably diagonal K3 surfaces. We then briefly discuss a variant based on a paper of Lang and Weil [**85**].

Since we describe an effective method, we need k and X given as explicit data. k may be given as a finite sequence of field extensions of $\mathbb{Q}$, where each field extensions is specified by an irreducible polynomial over the respective number field, of which adjoining a root yields the next field – the specifics are not important.

The Weil conjectures are only valid for smooth projective k-varieties. Because it is of general interest we also describe how to encode an arbitrary k-variety first before restricting to the projective case.

An affine k-variety X can be given by a finitely generated k-algebra R with $X = \mathrm{Spec}(R)$, and R can be given by a finite set of generators $\{x_1, \ldots, x_n\}$ and a finite set of relations among those generators $\{f_1, \ldots, f_m\}$ (over the noetherian ring k of finite type implies of finite presentation). Such a finite presentation of R gives rise to an embedding into affine n-space $\mathrm{Spec}(R) = X \hookrightarrow \mathbb{A}^n_k$. Note that a k-algebra R defined by such a representation does not automatically give rise to a k-variety in our sense: $\mathrm{Spec}(R)$ is of finite type over k, and it is separated, but in general neither reduced, nor irreducible, lest geometrically integral.

A k-variety X can be given by a finite set of affine k-varieties and patching data. The affine varieties can be given by a set $\{R_1, \ldots, R_r\}$ of finitely presented k-algebras as above. The patching data consists of r^2 isomorphisms of open subschemes of the $\mathrm{Spec}(R_i)$, concretely $\mathrm{Spec}(R_i) \supset U_{i,j} \xrightarrow{\phi_{i,j}\ \cong} U_{j,i} \subset \mathrm{Spec}(R_j)$. The $U_{j,i}$ can be specified by giving $\mathrm{Spec}(R_j) \setminus U_{j,i}$, which is a closed subscheme of $\mathrm{Spec}(R_j)$ and can be represented by finitely many elements $\{g_1, \ldots, g_s\}$ of the finitely presented k-algebra R_j that generate the associated ideal of $\mathrm{Spec}(R_j) \setminus U_{j,i}$ in R_j. Note that $U_{j,i}$ and $U_{i,j}$ can be assumed to be affine subschemes by [**61**, II. Exercise 4.3.], since we want them to represent the intersection of two affine, open subschemes of X, namely $\mathrm{Spec}(R_i) \cap \mathrm{Spec}(R_j)$. Then we get $U_{j,i} = \mathrm{Spec}(R_{j,i})$ where $R_{j,i}$ is a k-algebra derived from R_j by inverting a finite set of elements, e.g., $\{g_1, \ldots, g_s\}$ from above. Specifying $\phi_{i,j}$ amounts to specifying the images of $\tilde{x}_1, \ldots, \tilde{x}_n$, which are the associated elements of $R_{j,i}$ of the generators of R_j in the ring $R_{i,j}$, which is again a finite amount of data. So we can represent X by a finite amount of data. For this type of data to even represent a scheme certain compatibility conditions have to hold. See [**61**, II. Exercise 2.12.] for a proof that the ϕ are really isomorphisms and not just morphisms. If these conditions hold, then the k-scheme thus defined is of finite type, but not necessarily separated.

The compatibility conditions mentioned above, (geometrical) irreducibility and reducedness can be checked effectively, hence we can (in principle) recognize such data as giving a k-variety. If we would allow the $U_{i,j}$ to be non-affine, then we could describe all non-separated k-schemes of finite type, but the declaration of the morphisms $\phi_{i,j}$ would be more complicated, since we needed to cover the $U_{i,j}$ by open affine subschemes, define the isomorphism on each affine cover scheme, and ensure compatibility.

Given the technicalities of the description just introduced, we restrict ourselves to projective, more precisely projectively embedded, k-varieties. This does not restrict the generality of the approach as is showed in the paragraph after this useful lemma:

LEMMA 2.80 (Nishimura lemma). *Let l be a field and $f : X \dashrightarrow Y$ a rational map of l-schemes where X has a smooth l-point and Y is proper. Then Y has an l-point.*

PROOF. See, e.g., [**77**, Proposition 6] □

By Chow's Lemma (see [**61**, II Exercise 4.10.]) for a proper variety there is a projective k-variety and a morphism to the original proper variety that is an isomorphism between open dense subvarieties when suitably restricted. For each proper variety there is a birational map to a projective variety. Applying the Nishimura lemma and similar techniques outlined below for dealing with the singular locus, we can reduce decision of existence of $k_{\mathfrak{p}}$-rational points on proper varieties to projective varieties.

Also in order to apply the Weil bounds one needs projective varieties. We assume from now on that X is a projectively embedded k-variety, given by a set $\{f_1, \ldots, f_m\}$ of homogeneous polynomials in the polynomial ring $k[x_0, x_1, \ldots, x_n]$.

By a result of Nagata (see [**99**, Theorem 4.3.]) every k-variety embed as an open dense subvariety in a complete variety, but the Nishimura lemma now is only applicable in one direction. Hence the problem of deciding everywhere local solubility for non-proper varieties can not be solved by reduction to proper varieties via the Nishimura lemma only.

Assume for the moment that the projective variety X was not smooth. Since in characteristic 0, so in particular for the local fields $k_{\mathfrak{p}}$ associated to the number field k, we have resolution of singularities by Hironaka stated in proposition 2.32, there is a birational morphism $f : X' \to X$ with X' smooth. By composing any $k_{\mathfrak{p}}$-rational point $P \in X'(k_{\mathfrak{p}}) = \mathrm{Mor}_{\mathbf{Sch}}(\mathrm{Spec}(k_{\mathfrak{p}}), X')$ on X' with f, we get a $k_{\mathfrak{p}}$-rational point on X. Let g be the inverse rational map to the rational map induced by f. If X has a smooth $k_{\mathfrak{p}}$-rational point, then by the Nishimura lemma applied to g, X' has a $k_{\mathfrak{p}}$-rational point. This leaves the case when X has a non-smooth $k_{\mathfrak{p}}$-rational point. We investigate the singular locus $\mathrm{Sing}(X)$, which by [**61**, III. Lemma 10.5.] forms a closed subset, that can be endowed with the reduced induced subscheme structure, forming a reduced subscheme also denoted $\mathrm{Sing}(X)$ of smaller dimension than X. Looking at the finitely many irreducible components of $\mathrm{Sing}(X)$ one by one we can apply resolution of singularities, and a non-smooth $k_{\mathfrak{p}}$-rational point that corresponds to a smooth $k_{\mathfrak{p}}$-rational point on one of the irreducible components of $\mathrm{Sing}(X)$ induces a $k_{\mathfrak{p}}$-rational point on the respective Hironaka resolution. This leaves non-smooth $k_{\mathfrak{p}}$-rational point corresponding to non-smooth $k_{\mathfrak{p}}$-rational point on $\mathrm{Sing}(X)$. Since the dimension drops in each step, we may continue this process inductively and get a finite number of desingularizations of irreducible components of iterated singular loci, of which at least one needs to have $k_{\mathfrak{p}}$-rational point, if the original X did.

In summary a (possibly) non-smooth proper X has a $k_{\mathfrak{p}}$-rational point, if and only if one of certain finitely many smooth projective X_i does. Thus we may assume X to be a smooth projective variety over a number field k.

Next we state the Weil conjectures, which are concerned with varieties over finite fields.

PROPOSITION 2.81. *Let Y be a smooth projective variety of dimension n over a finite field $\mathbb{F}_q$, and define a function counting solutions over field extensions* *of $\mathbb{F}_q$ by $N_Y : \mathbb{N} \to \mathbb{N}, r \mapsto \#Y(\mathbb{F}_{q^r})$. The associated zeta function is given by the expression $Z_Y(t) := \exp(\sum_{r=1}^{\infty} N_Y(r)t^r/r)$ for those $t \in \mathbb{C}$ where the series converges.*

(1) *Z_Y coincides, where defined, with a rational function that is expressible as the quotient of polynomials with coefficients in $\mathbb{Q}$.*
(2) *Let E be the self-intersection number of the diagonal Δ_Y in $Y \times_{\mathrm{Spec}(\mathbb{F}_q)} Y$. Then Z_Y satisfies the following functional equation for an appropriate choice of the sign:*

$$Z_Y(1/(q^n t)) = \pm q^{nE/2} t^E Z_Y(t).$$

(3) *The analog of the Riemann hypothesis holds, that is:*

$$Z_Y(t) = \left(\prod_{\substack{i=0 \\ 2 \nmid i}}^{2n} P_i(t) \right) / \left(\prod_{\substack{i=0 \\ 2 \mid i}}^{2n} P_i(t) \right),$$

where $\forall i \in \{0, 1, \ldots, 2n\} : P_i \in \mathbb{Z}[t]$ with $P_i(t) = \prod_{j=1}^{m_i}(1 - \alpha_{i,j}t)$, where the $\alpha_{i,j}$ are algebraic integers of absolute value $|\alpha_{i,j}| = q^{i/2}$. Moreover $P_0 = (1 - t), P_{2n} = (1 - q^n t)$ independent of Y.
(4) *We call $B_i := \deg P_i$ the i-th Betti number. Then $E = \sum_{i=0}^{2n}(-1)^i B_i$. Assume Y was obtained by reduction modulo a prime ideal $\mathfrak{p}$ of a number ring $\mathfrak{o}_k$ from an arithmetic variety Y' over that ring. Denote the unique complex manifold obtained by $\times_{\mathfrak{o}_k}\mathbb{C}$ and GAGA theory by $\tilde{Y}$. Then the usual Betti numbers $\tilde{B}_i := \mathrm{rk}_{\mathbb{Z}-mod} H^i_{sing}(\tilde{Y}, \mathbb{Z})$ defined via singular cohomology (or any other of the many cohomology theories equivalent for smooth complex manifolds) satisfy $B_i = \tilde{B}_i$.*

Proof. The first complete proof was given by Deligne in [**31**] building on work of many mathematicians most notably Dwork, Grothendieck and Weil himself. There are several introductions to this subject, e.g., see [**61**, Appendix C], or the references mentioned in the introduction. □

Remark 2.82. The original proof uses an analogous version of the Lefschetz trace formula for the action of the Frobenius automorphism. The Frobenius automorphism F on $\mathbb{F}_q$ induces an action on the l-adic cohomology, i.e., for any prime l coprime to q, we get an induced action F_i^* on $H^i_l(Y) := \varprojlim_n H^i_{\acute{e}t}(Y, \mathbb{Z}/l^n\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. The polynomials P_i are given as characteristic polynomials of this action, concretely $P_i(t) = \det(\mathrm{id} - F_i^* t)$ where id and F_i^* are $\mathbb{Q}_l$-vector space endomorphisms on $H^i_l(Y)$. Thus the $\alpha_{i,j}$ are inverses of eigenvalues of vector space endomorphisms induced by the Frobenius automorphism.

With this remark we get the following important statement about point counting.

COROLLARY 2.83. *Let Y be a smooth projective variety of dimension n over a finite field $\mathbb{F}_q$ with Frobenius automorphism F. Then*

$$\#Y(\mathbb{F}_{q^r}) = \sum_{i=0}^{2n}(-1)^i(\sum_{j=1}^{m_i}\alpha_{i,j}^r) = \sum_{i=0}^{2n}(-1)^i(\mathrm{Tr}((F_i^*)^r)),$$

with Tr *the trace of a vector space endomorphism and $(F_i^*)^r$ the r-th power of such an endomorphism.*

PROOF. The first equality follows from the Weil conjectures, which relate the $\alpha_{i,j}$ to the zeta function of Y, which is up to the exponential a generating function related to the counting function N_Y. The rest is elementary but tedious calculation with formal power series. The second equality follows from the remark before and some linear algebra. □

Now we return to the smooth projective variety X defined by a finite set of homogeneous polynomials $\{f_1, \ldots, f_m\}$ over the number field k.

PROPOSITION 2.84. *There is a projective arithmetic variety $\mathfrak{X}$ over $\mathfrak{o}_k$ such that $\mathfrak{X}\times_{\mathfrak{o}_k} k \cong X$ via the canonical inclusion $\mathfrak{o}_k \hookrightarrow k$ (one says $\mathfrak{X}$ is an arithmetic model of X), and such that there is a finite set of non-archimedean places $S \subset^{finite} \Omega_k^0$ such that for all $\mathfrak{p} \in \Omega_k^0 \setminus S$ the scheme $\mathfrak{X} \times_{\mathfrak{o}_k} \kappa_\mathfrak{p} =: \mathfrak{X}_\mathfrak{p}$ via the canonical projection $\mathfrak{o}_k \twoheadrightarrow \kappa_\mathfrak{p}$ (the so called reduction modulo $\mathfrak{p}$ of the model) is a smooth projective $\kappa_\mathfrak{p}$-variety and the scheme $\mathfrak{X} \times_{\mathfrak{o}_k} \mathfrak{o}_\mathfrak{p} =: \mathfrak{X}_{\mathfrak{o}_\mathfrak{p}}$ via the canonical map $\mathfrak{o}_k \hookrightarrow \mathfrak{o}_\mathfrak{p}$ is in particular a smooth projective $\mathfrak{o}_\mathfrak{p}$-variety.*

PROOF. We may clear denominators of the coefficients of the f_i by multiplying the whole polynomial with them and get polynomials with coefficients in $\mathfrak{o}_k$ still defining X over k. These polynomials now define a projective scheme $\mathfrak{X}$ over $\mathfrak{o}_k$, which is clearly an arithmetic model of X. For a $\mathfrak{p} \in \Omega_k^0$ the reduction $\mathfrak{X}_\mathfrak{p}$ is smooth, if the Jacobi criterion for smoothness holds, i.e., the matrix $(\frac{\partial f_i}{\partial x_j})_{i=1,\ldots,m,\ j=0,\ldots,n}$ has rank $r := n - \dim(X)$ everywhere. This may be expressed by the $r \times r$-minors of this matrix, which are homogeneous polynomials $g_1, \ldots, g_s$: the ideal $I := \langle f_1, \ldots, f_m, g_1, \ldots, g_s\rangle$ contains 1. This can be checked over k doing Gröbner basis computations in finitely many steps, and it can be done in a way that avoids division of the coefficients in $\mathfrak{o}_k$. Since X is smooth, the Buchberger's algorithm produces some algebraic integer a. The amount of coefficients coming up in the execution of Buchberger's algorithm is finite. Each of these coefficients is contained only in finitely many prime ideals $\mathfrak{p}_k$ of $\mathfrak{o}_k$, and these make up the set S. Buchberger's algorithm is only sensitive to vanishing or non-vanishing of intermediate coefficients in its execution path. Thus over each ring $\mathfrak{o}_\mathfrak{p}$, respectively field $\kappa_\mathfrak{p}$, with $\mathfrak{p} \notin S$ the algorithm produces a considered in these rings, which is a unit. Therefore I contains 1 over these rings for primes avoiding S. □

REMARK 2.85. The proof gives an effective method to determine bounds for S in terms of their residue fields. Since there are complexity bounds for Gröbner basis algorithms over number rings and related problems like effective arithmetic Nullstellensatz (see [**81**]), there are a priori estimates on S depending only on the coefficients and the degree of the f_i. Their astronomical size makes them unusable in practice, and one resorts to apply more special properties of X and its defining polynomials. In most examples considered in the literature S turns out to be rather small, e.g., containing only several small primes and a few big ones, where small refers to the absolute value of the integral prime number over which they lay.

REMARK 2.86. A variety X over a number field k is said to have bad reduction at a non-archimedian place $\mathfrak{p}$, if and only if for no arithmetic model $\mathfrak{X}$ over $\mathfrak{o}_k$ the associated $\mathfrak{X}_\mathfrak{p}$ is smooth. Otherwise X is said to have good reduction at $\mathfrak{p}$.

REMARK 2.87. There are analogous versions of the last proposition in the situation when X is neither smooth, nor projective, nor separated, but then $\mathfrak{X}$ does not have these properties either. There are also intermediate versions. See [**73**, III. Lemma 3.4.].

Next we discuss Hensel's lemma and local solubility at a single finite place. Hensel's lemma comes in several variants, and we discuss them separately. There are yet more variants not discussed here. See [**94**, I.4.] for a version involving lifting a factorizations of a polynomial from $\kappa_\mathfrak{p}[x]$ to $\mathbb{Z}_\mathfrak{p}[x]$, and more generally the analog for complete base rings.

PROPOSITION 2.88. *Let* $h := (h_1, \dots, h_r) \subset \mathfrak{o}_\mathfrak{p}[x_1, \dots, x_n]$, $r \leq n$, *and* $a := (a_1, \dots, a_n) \in \mathfrak{o}_\mathfrak{p}^n$ *such that* $h(a) \equiv (0)_{i=i}^r \mod \mathfrak{p}$. *Let the Jacobian matrix* $J(h) := (\frac{\partial h_i}{\partial x_j})_{i=1,\dots,r,\ j=1,\dots,n}$ *satisfy* $J(h)(a)$ *has an* $r \times r$*-submatrix invertible over* $\mathfrak{o}_\mathfrak{p}$. *Then there is an element* $y := (y_1, \dots, y_n) \in \mathfrak{o}_\mathfrak{p}^n$ *with* $y \equiv a \mod \mathfrak{p}$ *and* $h(y) = 0 \in \mathfrak{o}_\mathfrak{p}^r$.

PROOF. This is a slightly modified special case of [**19**, III.4.5. Corollary 2.]. □

PROPOSITION 2.89. *Let* $h \in \mathfrak{o}_\mathfrak{p}[x_1, \dots, x_n]$ *and* $a := (a_1, \dots, a_n) \in \mathfrak{o}_\mathfrak{p}^n$ *such that there are integers* $m \in \mathbb{N}_0$ *and* $j \in \{1, \dots, n\}$ *with* $h(a) \equiv 0 \mod \mathfrak{p}^m$ *and* $v_\mathfrak{p}(\frac{\partial h}{\partial x_j}) =: k < m/2$. *Then there is an element* $y := (y_1, \dots, y_n) \in \mathfrak{o}_\mathfrak{p}^n$ *with* $y \equiv a \mod \mathfrak{p}^{m-k}$ *and* $h(y) = 0 \in \mathfrak{o}_\mathfrak{p}$.

PROOF. See [**114**, II.2.2. Theorem 1.] for a version for $\mathbb{Z}_p$; the general case is a consequence of [**11**, Theorem 1.1.3.]. □

PROPOSITION 2.90. *Let* R *be any commutative ring and* $I < R$ *a finitely generated ideal, let* $h := (h_1, \dots, h_r) \subset R[x_1, \dots, x_n]$ *and* $a := (a_1, \dots, a_n) \in R^n$ *such that* $h(a) \equiv 0^r \mod I$, *and assume that the Jacobian matrix* $J(h) := (\frac{\partial h_i}{\partial x_j})_{i=1,\dots,r,\ j=1,\dots,n} \in \mathrm{Mat}(R, r, n)$ *has evaluated at* a *a right-inverse* $W \in \mathrm{Mat}(R, n, r)$ *modulo* I, *i.e.,* $J(h)(a) \cdot W \equiv E_r \mod I$ *with* $E_r \in \mathrm{Mat}(R, r, r)$. *Then for all* $t \in \mathbb{N}$ *there is a* $y^t := (y_1^t, \dots, y_n^t) \in R^n$ *with* $y^t \equiv a \mod I$ *and* $h(y^t) \equiv 0 \mod I^t$, *where* I^t *is the* t*-th power of the ideal* I.

PROOF. See [**74**, 3.3.2. Satz]. □

REMARK 2.91. The condition of having a right-inverse can be replaced by the condition that certain linear systems of equation have solutions $\mod I$. E.g., when h contain duplicates, or when $r > n$, this may allow to apply the theorem despite $J(h)(a)$ having no right-inverse. The details are however rather technical but can be drawn immediately from the (constructive) proof in [**74**].

PROPOSITION 2.92. *Let X be a smooth projective k-variety with an arithmetic model such that for a given non-archimedian prime $\mathfrak{p}$ the schemes $\mathfrak{X}_{\mathfrak{o}_\mathfrak{p}}$ and $\mathfrak{X}_{\kappa_\mathfrak{p}}$ are smooth and projective. Assume that X has a $\kappa_\mathfrak{p}$-rational point a. Then X has a $\mathfrak{o}_\mathfrak{p}$-rational point y, and hence a $k_\mathfrak{p}$-rational point.*

PROOF. Let $f := (f_1, \dots, f_m) \subset \mathfrak{o}_\mathfrak{p}[x_1, \dots, x_n]$ define $\mathfrak{X}_{\mathfrak{o}_\mathfrak{p}}$. Let $a' \in X$ be the topological point which is the image of $a \in \mathrm{Mor}_{\mathbf{Sch}}(\mathrm{Spec}(\kappa_\mathfrak{p}), X)$. Since any smooth variety is a local complete intersection (see [**61**, II. Example 8.22.1.], which is easily seen to be true for arbitrary base fields not just for algebraically closed ones) we find an open affine neighborhood $a' \in U \subset X$ and $r = \mathrm{codim}(X, \mathbb{P}^n)$ many polynomials $h := (h_1, \dots, h_r) \subset f$ such that the arithmetic variety $\mathfrak{U}$ defined by f is isomorphic to U on some open subvariety of the generic fiber over $\mathrm{Spec}(k) \in \mathrm{Spec}(\mathfrak{o}_\mathfrak{p})$ and such that the special fiber over $\mathrm{Spec}(\kappa_\mathfrak{p}) \in \mathrm{Spec}(\mathfrak{o}_\mathfrak{p})$ is still smooth. Finding such an h is a standard task, achieved by linear algebra computation for the tangent space of X at a' defined over k and its reduction $\mod \mathfrak{p}$ defined over $\kappa_\mathfrak{p}$ via the differentials of the original f_i and their reductions. By assumption X is smooth at a' and has a smooth model over $\mathfrak{o}_\mathfrak{p}$, thus $J(h)(a')$ has an $r \times r$-submatrix invertible over $\mathfrak{o}_\mathfrak{p}$. In the terminology of proposition 2.90 with $R = \mathfrak{o}_\mathfrak{p}, I = \mathfrak{p}$, and h, a as indicated, $J(h)(a')$ has a right-inverse, and thus the existence of an $\mathfrak{o}_\mathfrak{p}$-rational point y follows. Alternatively we may use proposition 2.88. □

REMARK 2.93. The projective scheme Y defined by h contains X, but might have additional irreducible components and also behave bad otherwise. The twisted cubic (see [**61**, I. Exercises 1.2. and 2.17.]) is a good example. The important thing is that X and Y contain isomorphic smooth affine neighborhoods of a', since we only care about lifting the $\kappa_\mathfrak{p}$-rational point a at a' to an $\mathfrak{o}_\mathfrak{p}$-rational point.

REMARK 2.94. If X does not have a model whose reduction modulo $\mathfrak{p}$ is smooth, then the application of Hensel's lemma is technical and more difficult. One would need a generalization of both propositions 2.88 and 2.89. For hypersurfaces proposition 2.89 is applicable immediately. This is only necessary for the finitely many bad $\mathfrak{p}$. The first order theory over $\mathfrak{p}$-adic numbers is decidable as a consequence of quantifier elimination for such fields, worked out in a series of papers by Ax and Kochen [**8**] among other results regarding logical properties of such fields. This settles the problem of deciding local solubility at finitely many finite places in general. See also the lecture notes from the MSRI [**63**]. For a result with explicit bounds on the complexity of deciding solubility at a given prime for the ring of integers $\mathfrak{o}_\mathbb{Q} = \mathbb{Z}$, see a report of Chistov and Karpinski [**21**].

REMARK 2.95. In practice, the common type of examples are hypersurfaces. The well worked out method of Hensel lifting for single polynomials applies in that case. An example that goes beyond this is intersections of two hypersurface of certain low degrees, worked out by Wooley in [**133**].

REMARK 2.96. The variants of Hensel's lemma describe methods to get from a solution $\mod \mathfrak{p}^t$ the existence of a solution in $k_{\mathfrak{p}}$. But we also need criteria for non-existence of $k_{\mathfrak{p}}$-rational points.
To this end let X be given by a set of homogeneous equations $f := (f_1, \ldots, f_m)$ over $\mathfrak{o}_{\mathfrak{p}}$. If there is a $k_{\mathfrak{p}}$-point, then by clearing denominators, which is possible for homogeneous equations, there is an $\mathfrak{o}_{\mathfrak{p}}$-point. Let p be the integral prime number under $\mathfrak{p}$. Take a term $f_{i,j} \prod_{l=0}^{n} x_l^{\mu_l}$ of f_i, substitute p for all variables and denote its $\mathfrak{p}$-valuation by $v_{i,j}$. Let s_i be the number of terms of the polynomial f_i and denote $v := \max_{i \in \{1,\ldots,n\}} \{\max_{j \in \{1,\ldots,s_i\}} \{v_{i,j}\} + s_i\}$. A straightforward analysis shows that if X has an $\mathfrak{o}_{\mathfrak{p}}$-rational point, i.e., the f_i admit a nontrivial (meaning not all components 0) solution $a := (a_0, \ldots, a_n)$ of f in $\mathfrak{o}_{\mathfrak{p}}$, then the system of equation defined by f has a nontrivial solution $\mod \mathfrak{p}^v$: if each component of a had valuation $v_{\mathfrak{p}}$ at least v, then each component a_i would be divisible by p, easily seen from the construction of v, and thus we would get another nontrivial solution $a' := (a_0/p, \ldots, a_n/p)$. Repeating this, we would arrive at a nontrivial a'' of which at least one component had valuation less than v. Hence the reduction $\mod \mathfrak{p}^v$ of a'' would give a nontrivial element $\tilde{a}$ solving the system $f \equiv 0 \mod \mathfrak{p}^v$.
A finer analysis would allow to choose v even smaller. Another approach to reduce the size of v is to find polynomial systems f' equivalent to f that may allow for smaller v. See [**13**, Proposition 5.3.] for an explicit example, restated in proposition 2.102.

Next we deal with the infinite places.

REMARK 2.97. We consider the fields $\mathbb{R}$ and $\mathbb{C}$. Clearly a non-empty k-variety has $\mathbb{C}$-points. It only remains to check for $\mathbb{R}$-points for the various embeddings of $k \hookrightarrow \mathbb{R}$. This question is at the heart of real algebraic geometry and can be tested effectively. See the text book of Basu, Pollack and Roy [**10**]. The involved techniques are Sturm-chains, cylindrical algebraic decomposition, and Gröbner bases over $\mathbb{R}$.

REMARK 2.98. In most examples considered in practice, the real analysis is simpler. One either finds real solutions by inspection, or often finds an argument involving squares for non-existence.

Now we have all the ingredients to give a method for checking everywhere local solubility.

PROPOSITION 2.99. *Let X be a projective smooth variety over a number field k. The following method decides effectively everywhere local solubility for X.*

(1) *Determine a finite set of non-archimedian primes S' containing all the primes of bad reduction for X using, e.g., proposition 2.84.*
(2) *Determine the P_i, or more concretely the $\alpha_{i,j}$ of proposition 2.81, or at least the B_i for $i \in \{0, \ldots, 2\dim(X)\}$. Determine from this a lower bound b such that all $\mathfrak{X}_{\mathfrak{q}}$ with $\kappa_{\mathfrak{q}} \cong \mathbb{F}_q$ and $q \geq b$ must have an $\mathbb{F}_q$-point, provided $\mathfrak{X}_{\mathfrak{q}}$ is smooth, e.g., $\mathfrak{q}$ avoids S', using corollary 2.83. The finitely many non-archimedian primes with residue field containing less than b elements form the set S''.*
(3) *Set $S := S' \cup S''$. By proposition 2.92, which is a consequence of Hensel's lemma, we know that all places in $\Omega_k^0 \setminus S$ admit rational points. This leaves only the finitely many primes in S and $\Omega_k^{\mathbb{R}}$ to be checked.*
(4) *Test local solubility at the primes $\mathfrak{p}$ of S. For this either use variants of Hensel's lemma and determine a bound t' for the valuation as in proposition 2.89, and a bound t'', below which nontrivial solutions must exist, if there are local points as in remark 2.96, and then search for solutions* $\bmod \mathfrak{p}^t$, *where $t := \max\{t', t''\}$. Or relay on quantifier elimination and decidability of the first order theory of $k_{\mathfrak{p}}$, which might be less efficient, but guaranteed to work in general.*
(5) *Test local solubility at the places in $\Omega_k^{\mathbb{R}}$, e.g., as indicated in remark 2.97.*

If all tests of the last two steps are positive, then X is everywhere locally solvable. If one of these tests fails, it is not.

Proof. The only thing to discuss is probably the second step; everything else is clear from the references given in the statement. Correctness and effectivity of the method are immediate from the description.
The formula from corollary 2.83 was

$$\#X(\mathbb{F}_{q^r}) = \sum_{i=0}^{2n} (-1)^i (\sum_{j=1}^{m_i} \alpha_{i,j}^r) = \sum_{i=0}^{2n} (-1)^i (\mathrm{Tr}((F_i^*)^r)).$$

We determine the $\alpha_{i,j}$. When we know the B_i, this can be done by computing enough $\#Y(\mathbb{F}_{q^r})$ for varying q, r. We may also compute enough Frobenius actions on l-adic cohomology, which can be done effectively. With $|\alpha_{i,j}| = q^{i/2}$ and $P_{2n} = (1 - q^n t)$ forcing $\alpha_{2n,1} = q^n$, which dominates all the other $\alpha_{i,j}^r$ in the formula, we get $\#Y(\mathbb{F}_{q^r}) > 0$ for q, r big enough. One only needs good reduction, so we need that q is associated with a prime $\mathfrak{p}$ not contained in S'. Straightforward calculation with inequalities shows that a bound b is effectively established, as soon as one knows the $\alpha_{i,j}$. Accepting worse bounds the knowledge of only the B_i is enough, since the $\alpha_{i,j}$ are already bounded by their absolute value $|q^{i/2}|$. To determine the B_i, one does not need to resort to computing l-adic cohomology, since the Weil conjectures 2.81 relate them to the topological Betti numbers, which by GAGA theory are determined by the Hodge numbers for smooth projective varieties. Hodge numbers can be computed using ordinary sheaf cohomology of exterior powers of the sheaf of differentials. This is well-known to be effectively computable, e.g., using Cech cohomology. ☐

EXAMPLE 2.100. Let C be a smooth projective curve of geometric genus p_g over $\mathbb{Q}$. The Weil conjectures 2.81 yield $B_0 = B_2 = 1$. By definition of p_g and by Serre duality (see [**61**, III.7.]) we get $B_1 = 2p_q$. By the formula in corollary 2.83 we have $\#C(\mathbb{F}_q) = 1 - (\sum_{j=1}^{2p_g} \alpha_{1,j}) + q$, which with $|\alpha_{1,j}| = q^{1/2}$ yields the inequality $|\#C(\mathbb{F}_q) - 1 - q| \leq 2p_g\sqrt{q}$. Thus if C has good reduction mod q for q a prime and $q > 4p_g^2$, then C_q has an $\mathbb{Q}_q$-point.
Let C be a smooth plane curve given by an integral polynomial $f_1 := \sum_{i,j,l} f_{i,j,l}x_0^i x_1^j x_2^l$. The condition to be smooth can be expressed as $\sqrt{\langle f_1, \frac{\partial}{\partial x_0}f_1, \frac{\partial}{\partial x_1}f_1, \frac{\partial}{\partial x_2}f_1\rangle} = \langle x_0, x_1, x_2\rangle$. E.g., with the help of Gröbner basis, one finds 12 integral polynomials $g_{*,*}$, 3 integers A_* and another positive integer N with $g_{-1,j}f_1 + \sum_{i=0}^{2} g_{i,j}\frac{\partial}{\partial x_i}f_1 = A_j x_j^N$ for $j \in \{0,1,2\}$. The prime divisors of $A_0A_1A_2$ form a set of primes S' where C might have bad reduction. Thus we check local solubility explicitly only for the primes in S', primes smaller than $4p_g^2$ and the unique real place.
Similar considerations hold for smooth hyperelliptic curves, which have a birational model given by a degree 2 map to $\mathbb{P}^1$ specified by a polynomial $x_2^2 - h(x_1)$ with h either of degree $2p_g + 1$ or $2p_g + 2$.

EXAMPLE 2.101. Let X be a K3 surface over $\mathbb{Q}$. Proposition 2.44 tells us that $B_0 = B_4 = 1, B_1 = B_3 = 0$ and $B_2 \leq 22$, because the Hodge numbers can only increase after base change to the algebraic closure. The Weil bounds give $|\#X(\mathbb{F}_q) - 1 - q^2| \leq 22q$. So there is an $\mathbb{Q}_q$-point for every prime $q \geq 23$ where X has good reduction.

PROPOSITION 2.102. *Let X be a diagonal quartic surface in $\mathbb{P}^3_{\mathbb{Q}}$ defined by a polynomial with coprime quarticfree integral coefficients $f = a_0x_0^4 + a_1x_1^4 + a_2x_2^4 + a_3x_3^4$. Let S' be the set of prime devisors of $a_0a_1a_2a_3$. Then X has p-adic points at all p except possibly at $S := S' \cup \{2,5\}$.*
Let p be an odd prime, e.g., in S. For $i \in \{0,1,2,3\}$ let f_i be the polynomial obtained from f by multiplying each coefficient with p^i and then dividing the new coefficients by possible p^4-factors individually. Then X has p-adic points, if and only if for some $i \in \{0,1,2,3\}$ the variety X_i defined by the f_i has smooth $\mathbb{F}_p$-points.
Let $p = 2$ and define the f_i in an analogous way. Then X has a 2-adic point, if and only if for some $i \in \{0,1,2,3\}$ the variety X_i has a $\mathbb{Z}/2^5\mathbb{Z}$-point with coordinates $(y_0 : y_1 : y_2 : y_3)$ such that $2 \nmid a_jy_j$ for some $j \in \{0,1,2,3\}$.
X has a real point if and only if not all coefficients of f have the same sign.

PROOF. See Bright's thesis [**13**, 5.1.]. We give a sketch.
The primes of bad reduction are contained in S' and $\{2\}$ as the Jacobi criterion tells us. By the last example there are p-adic points for any other prime $p \geq 23$. The primes in $\{3,7,11,13,17,19\}$ give rise to p-adic points either by case by case

computations, or by a short, more clever argument as Bright did. This leaves the primes in S.
For the odd primes one direction is an immediate application of Hensel's lemma as in proposition 2.89. The other direction involves arguments similar to the ones outlined in remark 2.96.
The argument for the prime 2 works along the same lines, but applying Hensel's lemma is more difficult, since deriving f_i introduces additional powers of 2.
The real case is clear, because all variables appear with even powers in f. □

Instead of using the Weil conjectures to get lower bounds on the size of finite fields from where on one must have rational points on the variety for that field, one can also use a conceptually much simpler approach, which goes back to Weil and Lang in [**85**]. The Weil conjectures and therefore the Weil bounds for curves as discussed in example 2.100 are – compared to the general case – relatively simple. They were proved by Weil himself in the 40s. The idea of the paper [**85**] is to inductively cut out hyperplane sections starting from a variety of fixed dimension (r), degree (d) and codimension ($n-r$) relative to a fixed projective embedding, and to control these quantities for the inductively generated subvarieties. In this step one uses an effective variant of Bertini's theorem (cf. [**61**, III. Corollary 10.3.]) to ensure that the hyperplane sections stay varieties in a number of cases bounded from below. Finally when dimension is down to one one uses the Weil bounds for curves. This method also leads to effectively computable constants $A(n,d,r)$ with

$$|\#Y(\mathbb{F}_{q^r}) - q^r| = (d-1)(d-2)q^{r-1/2} + A(n,d,r)q^{r-1}.$$

These results inspired work on decidability of first order theory over fields, such as the results of Ax and Kochen [**8**] mentioned above, and further developments on the logical behavior of more general classes of fields (see [**41**, pp. xx/xxi]).

The approach of Lang and Weil to reduce dimension inductively and control invariants that ultimately control the number of $\mathbb{F}_q$-points leads immediately to another method, which can be exploited in examples, where one has more information on the (type of) variety: instead of artificially creating subvarieties defined over the base field by hyperplane sections, there might be some obvious candidates such as fibers of a fibration, or knowledge of embedded curves defined over the base field in a given variety. One looses to have bounds for the number of $\mathbb{F}_q$-rational points in both direction, but a lower bound on the number of such points of the embedded variety gives a lower bound for them on the whole original variety.

Embedded lower dimensional subvarieties that we already know of are usually easier to analyse than forcing such subvarieties by hyperplane sections, and one can also hope for better bounds.

With this remarks we finish our account on everywhere local solubility.

2.3. Miscellenea

2.3.1. Fibrations of Surfaces over Number Fields

DEFINITION 2.103. Let k be a field and X a smooth projective k-surface. We call a proper connected dominant k-morphism $f : X \to C$ to a smooth projective k-curve a fibration and X a fibered surface via f.

REMARK 2.104. The notion of elliptic fibration of definition 2.36 for surfaces over an algebraically closed field generalizes immediately. Since the general fiber is an elliptic curve, and thus of smaller dimension than the connected X, and the base is 1-dimensional, any elliptic fibration must be dominant. Therefore it is a special case of a fibered surface, and we use this term now for arbitrary base fields.

REMARK 2.105. By [**61**, III. Proposition 9.7.] any fibration in our sense is flat. If one wants to generalize the notion of fibration to higher dimensional bases or more complicated schemes than surfaces, one needs to require flat in addition to make the analogous theory work.

REMARK 2.106. We call the pullback of a fibration to an open subcurve of the base a fibration. We also call the morphism induced by a fibration on the complement of a closed subscheme of a fibered surface a fibration. In chapter 3 however we mean by a fibration exactly the notion of definition 2.103.

REMARK 2.107. We call a finite flat surjective morphism of varieties $f : X \to Y$ a cover (covering).

PROPOSITION 2.108. *Let $f : X \to C$ define a fibered surface. Let $\eta \hookrightarrow C$ be the generic point and $X_\eta := X \times_C \eta \hookrightarrow X$ the generic fiber of f. Then the induced homomorphism* $\mathrm{Pic}(X) \xrightarrow{\phi} \mathrm{Pic}(X_\eta)$ *is surjective, and* $\ker(\phi)$ *is generated by the classes of the irreducible components of the fibers of f.*

PROOF. See [**13**, Proposition 2.14.]. The proof is a standard exercise in sheaf cohomology for the Zariski topology. □

DEFINITION 2.109. For a fibration f as in the last proposition, we call $\mathrm{Pic}_{vert}(X) := \mathrm{Pic}_{vert,f}(X) := \ker(\phi)$ the vertical Picard group for the fibration f.

REMARK 2.110. Any smooth projective surface X over an algebraically closed field k admits a birational morphism from a smooth projective surface $b : Y \to X$ obtained by blowing up a finite number of points such that Y admits a Lefschetz fibration, i.e., a flat fibration $f : Y \to \mathbb{P}^1$, such that f has a section, the generic fiber is a smooth curve, and any fiber has at most a single node singularity. See [**94**, V. Theorem 3.1.].
This means that fibrations with relatively nice properties are quite common. One usually also wants in addition to control the general fiber, e.g., elliptic fibrations. This is harder and not always possible, but would give an explicit description of the Picard group, which comes in handy very often. For number fields the problem

remains to descent a fibration over the algebraic closure to the original field. Over algebraically closed fields k of characteristic 0 any smooth projective k-variety of arbitrary dimension admits a Lefschetz fibration, where the notion of (Lefschetz) fibration is generalized in a straightforward manner to higher dimension. One can prove this, e.g., by reducing to the complex case and then doing a simple argument about generic behavior of hyperplane sections in complex analytic geometry. See [**125**, Corollary 6.1.2] for an explicit reference.

REMARK 2.111. Usually one is only interested in the existence of fibrations up to birational equivalence of smooth projective surfaces. The Brauer group is an invariant under this equivalence (see proposition 3.80 and corollary 3.96), which is our main application of fibrations. This justifies calling a rational map $f : X \dashrightarrow C$ a fibration, when the induced morphism after resolving the locus of indeterminancy of f (see [**61**, II. Example 7.17.3.]) is a fibration $f' : X' \to C$.

DEFINITION 2.112. Let $f : X \to C$ be a fibration over some field k. We say that a closed k-subscheme $M \subset X$ is a multisection of degree d for f if and only if the induced morphism $f' : M \hookrightarrow X \to C$ is a finite morphism of degree d.

REMARK 2.113. Let $f : X \to C$ be a fibration. Using the notation of proposition 2.108 we take a closed point μ of the generic fiber X_η and take its closure $M := \overline{\mu} \subset X$, which is a 1-dimensional closed subscheme. The induced map $M \xrightarrow{f'} C$ is a dominant morphism between integral 1-dimensional projective schemes, and hence finite, therefore it defines a multisection of degree $\deg(f')$.

Now we collect some results about diagonal quartics and their fibrations taken mostly from [**13**].

PROPOSITION 2.114. *Let $k = k^{alg}$ be an algebraically closed field of characteristic 0 and X a diagonal quartic. We may assume that X is given by the equation $x_0^4 - x_1^4 = x_2^4 - x_3^4$ in $\mathbb{P}^3_k$. The following rational maps equip X with a fibration:*

(1) $X \dashrightarrow \mathbb{P}^1, [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 - x_1 : x_2 - x_3]$,
(2) $X \dashrightarrow \mathbb{P}^1, [x_0 : x_1 : x_2 : x_3] \mapsto [x_0^2 - x_1^2 : x_2^2 - x_3^2]$,
(3) $X \dashrightarrow \mathbb{P}^1, [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : x_1]$.

The first two fibrations are elliptic fibrations. The last is a fibration in curves of geometric genus 3.
Let $\alpha, \beta \in \mu_4$, where μ_4 is the set of 4-th roots of unity, and ζ_8 is an 8-th root of unity. Then $\mathrm{Pic}(X) \cong \mathbb{Z}^{20}$ is generated by the classes of the 48 lines defined by the following pairs of equation:

$$\{x_0 = \alpha x_1, x_2 = \beta x_3\}, \{x_0 = \alpha x_2, x_1 = \beta x_3\}, \{x_0 = \alpha\zeta_8 x_3, x_1 = \beta\zeta_8 x_2\}.$$

We have $\mathrm{Pic}(X) \cong \mathrm{NS}(X)$, and both carry a lattice structure by the intersection pairing.

PROOF. See [**13**, Proposition 2.15. and p. 23] and proposition 2.44, which proves everything except the statements about the first and last fibration.
The fibers of the first rational map are hyperplane sections in $\mathbb{P}^3$ and all contain the line defined by the system $\{x_0 - x_1 = 0 = x_2 - x_3\}$. This fibration is a projection from one of the 48 lines on a diagonal quartic. A hyperplane section of a quartic surface gives a plane quartic curve, and therefore the complement of the line in this plane is a plane cubic, which is an elliptic curve, if it does not degenerate. If a single fiber does not degenerate, then the general fiber does not, too. This is easily checked at a concrete fiber, e.g., the one at $[1 : 0]$.
The fibers of the third rational map can in a similar way be shown to be non-degenerate plane quartics. The claim follows from the well-known genus formula for smooth plane degree d curves $p_g = \frac{(d-1)(d-2)}{2}$ (see [**61**, II. Exercise 8.4.(f)]). $\square$

PROPOSITION 2.115. *Let $k = \mathbb{Q}$ and X a diagonal quartic given by an equation as in proposition 2.102. Then X has an elliptic fibration $X \dashrightarrow \mathbb{P}^1_{\mathbb{Q}}$ over $\mathbb{Q}$, if and only if the restricted intersection form on $\mathrm{Pic}(X^{alg})^{\mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})}$ represents 0 nontrivially. Let M be a representing matrix for the intersection form on $\mathrm{Pic}(X^{alg})^{\mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})}$, and define $\rho := \mathrm{rk}\,\mathrm{Pic}(X^{alg})^{\mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})}$.*

(1) *If $\rho = 1$, then X does never admit an elliptic fibration over $\mathbb{Q}$,*
(2) *if $\rho = 2$, then X does admit an elliptic fibration over $\mathbb{Q}$, if and only if $(-\det(M))$ is a square, i.e., $-\det(M) \in \mathbb{Q}^{(2)}$,*
(3) *and if $\rho \geq 3$, then X does always admit an elliptic fibration over $\mathbb{Q}$.*

When X is everywhere locally solvable then $\mathrm{Pic}(X^{alg})^{\mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})} \cong \mathrm{Pic}(X)$.

PROOF. See [**13**, Proposition 2.23. and Corollary 2.27] and proposition 3.119 from the next chapter. $\square$

2.3.2. Motivation for Diagonal Quartics

At the end of the chapter on arithmetic geometry, we give some reasons why diagonal quartics are an interesting test case.

As described in subsection 2.1.3, the arithmetic of curves is qualitatively well understood: Kodaira dimension, respectively ampleness of the canonical invertible sheaf, govern it.

The natural next case is surfaces: does ampleness of the canonical invertible sheaf or the Kodaira dimension also govern the behavior of the arithmetic? This turns out to be wrong, but the classification of surfaces seems to give at least an indication of what to expect.

According to Abramovich in [**1**, 1.7.], the Kodaira dimension $-\infty$ case is well understood. For the case of surfaces with an ample anti-canonical invertible sheaf (Fano surfaces = del Pezzo surfaces), which are automatically birational to $\mathbb{P}^2$, one has even a conjecture on the asymptotic growth rate of the rational points, namely the Manin conjecture (or Batyrev-Manin conjecture, which was a more general first attempt, that turned out to be wrong), which is verified in many cases and is currently actively research.

The next case is Kodaira dimension 0. Rational points on abelian varieties are qualitatively described by the Mordell-Weil theorem, and a generalization of the Birch and Swinnerton-Dyer conjecture is widely believed to describe it quantitatively. Bi-elliptic surfaces are finite quotients of products of elliptic curves, and Enriques surfaces are quotients by a $\mathbb{Z}/2\mathbb{Z}$-action of a K3 surface. To understand these surfaces, one should try to understand their coverings. This brings us to K3 surfaces, which are the genuine new phenomenon in Kodaira dimension 0, when going from curves to surfaces. In the endeavor to understand the qualitative behavior of arithmetic on varieties in low dimension, they are therefore the next choice for investigation.

Kodaira dimension 1 is also being currently researched, but the general picture is not clear. It should depend on how bad K3 surfaces behave as lower Kodaira dimension generally is associated with less complexity. General surfaces of Kodaira dimension 2 seem to be too difficult at the moment. However the Bombieri-Lang conjecture predicts that rational points should never be Zariski dense in this case, although the only thing known about it seems to be in connection to results on the Mordell-Lang conjecture for abelian varieties, i.e., not much.

There are also many other topics related to questions on rational points on surfaces, like potential density, curves on varieties over number fields, point counting with respect to heights, local densities and Tamagawa numbers, that we just mention for completness, but do not discuss any further.

K3 surfaces are also interesting because they seem to be a good test site for the power of the Brauer-Manin obstruction. The leading experts do not agree, whether one should expect the Brauer-Manin obstruction to existence of rational points to be the only one, or not. One may think about results on K3 surfaces as enriching our knowledge about a much more general method, namely the Brauer-Manin obstruction.

It has already been discussed in subsection 2.1.5 that the simplest way to study K3 surfaces is to study them in low degree, most prominently double covers of the plane branched in a sextic, or quartic surfaces. Diagonal quartics, which are a very special case of quartic surfaces, are the easiest case in that class, and the advantage of being able to do explicit computation outweighs the loss of generality.

See the section on the behavior of the Brauer group and the Brauer-Manin obstruction for surfaces 3.4 for further results.

CHAPTER 3

Brauer Groups and Brauer-Manin Obstruction

The standard reference for Brauer groups over schemes are the three articles of Grothendieck "le group de brauer I-III" in [**49**]. The book of Milne [**94**] also features chapter IV "the brauer group" – any scheme is assumed to be locally noetherian throughout the book without further mentioning. For Brauer groups over fields the recent book [**47**] of Gille and Szamuely gives a detailed introduction.

The Brauer-Manin obstruction was discovered by Manin and first described in [**88**] and [**89**, ch. VI]. One of the many newer basic introductions written in a colloquial style is [**56**] by Gounelas. A more abstract one can be found in [**120**] of Skorobogatov.

3.1. The Brauer Group

Historically the Brauer group was first defined for fields by Brauer and studied in the 1920s by him and other prominent algebraists of the time. Azumaya (1951) and Auslander and Goldman (1960) generalized this concept to rings and Grothendieck in [**49**] generalized it further to locally ringed topoi. We give the definition for schemes. We also define them separately for rings and fields, since the standard terminology for these special cases differ a little bit. Up to isomorphism the definitions agree.

3.1.1. The Azumaya Brauer Group

The statements of this subsection are all standard results. We mostly refer to the literature for proofs.

Definition 3.1. Let k be a field. A central simple k-algebra (k-csa) A is a k-algebra (see 1.1)

(1) that is of finite dimension viewed as a k-vector space,
(2) that is simple, i.e., the set of two-sided ideals of A is $\{\{0\}, A\}$, and the multiplication in A is not the 0-map, in particular A is not the 0-ring,
(3) and such that the center of A is k, i.e., $\{x \in A : \forall a \in A : ax = xa\} = k$, where we identify k with its image via the algebra structure map $k \to A$, that is always an inclusion, since k is a field.

Definition 3.2. Let k be a field and A, B two k-csa. We call A and B equivalent (Morita-equivalent, Brauer-equivalent), denoted $A \sim B$, if and only if

$$\exists n, m \in \mathbb{N} : A \otimes_k \mathrm{Mat}(k; n, n) \cong B \otimes_k \mathrm{Mat}(k; m, m).$$

The algebra $\mathrm{Mat}(k;n,n)$ is the full matrix algebra over k of quadratic $n \times n$-matrices, and the isomorphism is in the category of k-algebras.

For the following statement we use Grothendieck universes (see 1.1).

PROPOSITION 3.3. *Let k be a field contained in a given universe U. Brauer equivalence defines an equivalence relation on the set of all k-csa that are elements of U, which is coarser than the isomorphism relation. The quotient set by this relation is U-small. We write the class of a k-csa $A \in U$ as $[A] \in \mathrm{Br}(k)$.*
The tensor product of k-algebras induces a map

$$\cdot : \mathrm{Br}(k) \times \mathrm{Br}(k) \to \mathrm{Br}(k), ([A],[B]) \mapsto [A] \cdot [B] := [A \otimes B],$$

which turns out to be a group law on $\mathrm{Br}(k)$. The neutral element is $[k]$, and the inverse map is

$$\bullet^{op} = \bullet^{-1} : \mathrm{Br}(k) \to \mathrm{Br}(k), [A] \mapsto [A^{op}],$$

where for a k-csa A with ring structure $(A, 0, 1, +, \cdot : A \times A \to A, (x,y) \mapsto x \cdot y)$, we define the opposite algebra A^{op} by the ring structure $(A, 0, 1, +, \cdot^{op} : A \times A \to A, (x,y) \mapsto y \cdot x)$ and the same k-module structure as A. If $A \in U$, then $A^{op} \in U$. This group is a torsion group.
If U and V are two universes containing k, then the map between the two associated Brauer groups defined by sending a U-Brauer equivalence class $[A]$ to the unique V-Brauer equivalence class $[B]$ that satisfies $[A] \cap [B] \neq \emptyset$ is a group isomorphism.

PROOF. See [**47**, Prop 2.4.8.]. The torsion group statement is implicit in this work. $\square$

DEFINITION 3.4. Let k be a field. With the above terminology the group $(\mathrm{Br}(k), [k], \cdot)$ for some universe U containing k is called the Brauer group of k. By 3.3 the choice of U is essentially unimportant and is therefore not touched upon in what follows.

REMARK 3.5. One can avoid the additional assumptions needed for Grothendieck universes, if one works with chosen representations for k-csa, e.g., pairs of tupel vector spaces and k-algebra structure constants. To prevent clumsy notation we decided to use universes.

PROPOSITION 3.6. *Let X be a scheme and $\mathcal{A}$ a sheaf of $\mathcal{O}_X$-algebras, i.e., a sheaf of rings over X together with a morphism of sheaves of rings $\mathcal{O}_X \to \mathcal{A}$. Assume $\mathcal{A}$ to be of finite presentation when viewed as an $\mathcal{O}_X$-module via the algebra structure morphism. Then the following are equivalent:*

(1) *$\mathcal{A}$ is locally free as an $\mathcal{O}_X$-module, and for all $x \in X$ the algebra $\mathcal{A}(x) := \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is a $k(x)$-csa.*
(2) *For all $x \in X$ there are an $r \in \mathbb{N}$, an open neighborhood $x \in U \subset X$ and a finite étale surjective morphism $f : U' \to U$ such that $f^*\mathcal{A}|_{\mathcal{U}} \cong \underline{\mathrm{Mat}}(O_{U'}; r, r)$ as sheaves of $\mathcal{O}_{U'}$-algebras, where $\underline{\mathrm{Mat}}(O_{U'}; r, r)$ is the sheaf of $r \times r$-matrices.*

PROOF. See [**49**, le group de Brauer I, Théoèm 5.1.]. □

DEFINITION 3.7. Let X be a scheme. A sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ satisfying the equivalent conditions of the last proposition is called a sheaf of Azumaya algebras on X (X-sAa).

DEFINITION 3.8. Let X be a scheme and $\mathcal{A}, \mathcal{B}$ two X-sAa. $\mathcal{A}$ and $\mathcal{B}$ are called equivalent (Morita-equivalent, Brauer-equivalent), denoted $\mathcal{A} \sim \mathcal{B}$, if and only if there are locally free nontrivial sheaves of $\mathcal{O}_X$-modules $\mathcal{E}, \mathcal{E}'$ of everywhere finite but not necessarily constant rank such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}(\mathcal{E}) \cong \mathcal{B} \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}(\mathcal{E}')$$

as sheaves of $\mathcal{O}_X$-algebras, where $\underline{\mathrm{End}}(\mathcal{E}), \underline{\mathrm{End}}(\mathcal{E})'$ are the sheaves of endomorphism algebras of $\mathcal{E}$ respectively $\mathcal{E}'$.

For the following statement we disregard the set theoretical problems. They can be treated as in Proposition 3.3. We continue this omission for the rest of this work.

PROPOSITION 3.9. *Let X be a scheme. Brauer equivalence defines an equivalence relation on the set of all X-sAa. The set of all Brauer equivalence classes is denoted by* $\mathrm{Br}(X)$*. Denote the equivalence class of an X-sAa $\mathcal{A}$ by $[\mathcal{A}]$. The tensor product of sheaves of $\mathcal{O}_X$-algebras induces a well defined map on the equivalence classes*

$$\cdot : \mathrm{Br}(X) \times \mathrm{Br}(X) \to \mathrm{Br}(X), ([\mathcal{A}], [\mathcal{B}]) \mapsto [\mathcal{A}] \cdot [\mathcal{B}] := [\mathcal{A} \otimes \mathcal{B}],$$

which turns out to be a group law on $\mathrm{Br}(X)$ *with neutral element* $[\mathcal{O}_X]$*. The inverse map is constructed from the sheaf of opposite algebras $\mathcal{A}^{op}$ as in proposition 3.3. This group is a torsion group, if X has only finitely many connected components.*

PROOF. See [**49**, le group de Brauer I, 1., 2. & 5.]. □

DEFINITION 3.10. Let X be a scheme. With the above terminology the group $(\mathrm{Br}(X), [\mathcal{O}_X], \cdot)$ is called the Brauer group (Azumaya Brauer group) of X.

DEFINITION 3.11. Let R be a ring and A an R-algebra. The sheaf of modules $\mathcal{A} := \tilde{A}$ on $\mathrm{Spec}(R)$ can be endowed with the structure of a sheaf of algebras $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ induced by the multiplication on A. Then A is called Azumaya or R-Azumaya algebra (R-Aa), if and only if $\mathcal{A}$ is a sheaf of Azumaya algebras on $\mathrm{Spec}(R)$. $\mathcal{A}$ is called the associated sheaf of A.

PROPOSITION 3.12. *Let R be a ring. Then for any* $\mathrm{Spec}(R)$*-sAa $\mathcal{A}$ there is an R-algebra A such that $\mathcal{A}$ is isomorphic to the associated sheaf of A.*

PROOF. By [**61**, II. Exercise 5.4.] a locally free sheaf of modules is quasi-coherent. By [**61**, II. Corollary 5.5.] we have an equivalence of the category of quasi-coherent sheaves of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules and the category of R-modules. This yields forgetting the algebra structure for a moment an R-module A such that the sheaf $\mathcal{A}$ of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules is isomorphic to $\tilde{A}$. This implies in particular $\mathcal{A}(\mathrm{Spec}(R)) \cong A$ as R-modules. Since $\mathcal{A}$ is actually a $\mathrm{Spec}(R)$-sAa, the $\mathcal{A}(\mathrm{Spec}(R))$ carries the structure of an $\mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) = R$-algebra, which via the previous isomorphism endows an R-algebra structure on A. By the sheaf properties a sheaf is determined by sections on an open cover, and since $\mathrm{Spec}(R)$ is clearly an open cover, we have an isomorphisms of sheaves of $\mathcal{O}_{\mathrm{Spec}(R)}$-algebras $\mathcal{A} \cong \tilde{A}$. □

Since Brauer equivalence clearly respects isomorphisms, this proposition motivates the following definition.

Definition 3.13. Let R be a ring. The Brauer group of R is $\mathrm{Br}(R) := \mathrm{Br}(\mathrm{Spec}(R))$.

Corollary 3.14. *Let k be a field. Due to 3.6 a k-Aa as in 3.11 is a k-csa as in 3.4 and vice versa. Since sheaves of endomorphism algebras of locally free sheaves of modules on points like* $\mathrm{Spec}(k)$ *are easily seen to be equivalent to rings of k-matrix algebras, two k-Aa are equivalent, if and only if they are equivalent as k-csa. Since the equivalence between k-Aa and k-csa is actually one of monoidal categories (with respect to the standard $\otimes$-product in each category), the group laws defined by the $\otimes$ are compatible, and* $\mathrm{Br}(k)$ *in the k-Aa sense is naturally isomorphic to* $\mathrm{Br}(k)$ *in the k-csa sense.*
This justifies our abuse of notation to write only $\mathrm{Br}(k)$*, to denote both groups.*

Remark 3.15. The condition "of finite presentation" is a local notion for sheaves (see [**55**, 0. (5.2.5)]). The resulting Brauer group is therefore in general not torsion, cf. [**49**, I, Corollaire 1.5.]. Assuming X to have only finitely many connected components ensures the Brauer group to be torsion, see [**94**, IV. Proposition 2.7.]. An explicit example can be constructed from the field of p-adic numbers $\mathbb{Q}_p$ where p is an arbitrary prime. It is a fact (see the discussion before 3.109) that $\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$, which implies that for each $n \in \mathbb{N}$ there is a $\mathbb{Q}_p$-csa A_n, whose class in $Br(\mathbb{Q}_p)$ has order n. Let $A := \bigoplus_{n\in\mathbb{N}} A_n$ be an Azumaya algebra on $R := \bigoplus_{n\in\mathbb{N}} \mathbb{Q}_p$. It induces a non-torsion element in $\mathrm{Br}(R)$.
In the light of subsection 3.1.2 there are also schemes with infinitely many connected components and torsion, even trivial, Brauer group, e.g., $\mathrm{Spec}(\bigoplus_{n\in\mathbb{N}} \mathbb{C})$.

3.1.2. Examples of csas and Brauer Groups

This section is concerned with giving some examples of k-csas for various fields k and their Brauer groups. Generalizations of these examples to rings or schemes are not considered, and any statement given may or may not generalize to that setting. For further examples and results for concrete classes of varieties, see [**73**, II.8.]

Example 3.16. The Hamiltonian quaternions $\mathbb{H}$ are a $\mathbb{R}$-csa not Brauer equivalent to $\mathbb{R}$. See almost any text book on the subject, e.g., [**75**, 0.].

EXAMPLE 3.17. Let K/k be a cyclic Galois extension with Galois group $\langle\sigma\rangle$ and let $a \in K^* := K \setminus \{0_K\}$. For $n \in \mathbb{N}_{>1}$ consider $A := K^n$ the k-vector space with standard basis $(e_i)_{i=1}^n$. Choose the embedding $K \hookrightarrow A, b \mapsto be_1$. There is a unique k-algebra structure on A extending the multiplication of K induced by the embedding that satisfies $e_2^n = ae_1$ and $\forall b \in K : e_2 b = b^\sigma e_2$. This is a k-csa denoted by (K, σ, a). k-csas isomorphic to one of this type are called cyclic. See [**75**, 10.3 Satz], [**47**, 2.5] or [**103**, 15.1.].
Let l be a non-archimedian local field. For $n \in \mathbb{N}$ let l_n be the unique unramfied extension of l of degree n, denote by F_n its Frobenius morphism and by π its uniformizer (see [**100**] or [**42**]). Then for $m \in \mathbb{Z}$ we have a l-csa (l_n, F_n, π^m). To avoid trivialities set $(l_1, F_1, \pi^m) := k$

REMARK 3.18. There is also the more general crossed product construction for csas over arbitrary fields, which is essentially based on 2-cocycles (see [**75**, 7.] or [**103**, 14.1.]), of which cyclic csas are a special case. In general there are crossed product csas that are not cyclic ([**103**, 15.7. and Notes on Chapter 15]) and even non-crossed product csas ([**103**, 20.8.]).
For any field each Brauer equivalence class is representable by a crossed product csa (consequence of [**103**, 13.5. Corollary and 14.2. Theorem]). If k is a local or a global field (cf. subsection 2.2.1), then every k-csa (not just its Brauer equivalence class) is cyclic (see [**101**, (8.1.14) Proposition] for a coarser statement about Brauer groups, or [**103**, Preamble of 15.]).

EXAMPLE 3.19. We have $\mathrm{Br}(\mathbb{R}) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ generated by $[\mathbb{H}]$ and $\mathrm{Br}(\mathbb{C}) \cong 0 \cong \mathbb{Z}/\mathbb{Z}$. See [**75**, 0. Problem]. The unusual notation for this group is on purpose to make it compatible with definition 3.109.

EXAMPLE 3.20. More generally for $k = k^{alg}$ an algebraically closed field it holds that $\mathrm{Br}(k) \cong 0$. See [**75**, 3.6 Satz].

EXAMPLE 3.21. Every C_1-field (QAC-field) k has $\mathrm{Br}(k) \cong 0$, see [**117**, IV.3. Corollary 1]. Algebraically closed fields are C_1-fields([**117**, IV.3. Definition 6: Remarks (1)]), function fields of curves over algebraically closed fields are C_1-fields ([**117**, IV.3. Theorem 24]), which is a version of Tsen's theorem, finite fields are C_1-fields ([**117**, IV.3. Theorem 25]), which generalizes Wedderburn's little theorem, and some other interesting classes of fields are also C_1-fields ([**117**, IV.3. Theorem 27], Lang's theorem).

EXAMPLE 3.22. Let l be a non-archimedian local field, e.g., $\mathbb{Q}_p$. Then $\mathrm{Br}(l) \cong \mathbb{Q}/\mathbb{Z}$. See [**75**, 13.10 Satz], or the discussion before definition 3.109.

3.1.3. The Cohomological Brauer Group

We encounter several cohomology theories in the following. In the special cases we consider some or even most of these theories give isomorphic groups. Often in the literature they are not distinguished. For the sake of a clearer exposition, we distinguish them by subscripts introduced now. In [**94**] one finds the general concept of cohomology theories for sites, and the book of Tamme [**124**] is, as the title suggests, an introduction to étale cohomology, which is concise and well written. We suppose familiarity with homological algebra in the following.

Let X be a scheme. For $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules let $H^i_{zar}(X,\mathcal{F})$ be the derived functor sheaf cohomology for the Zariski site as in [**61**, III.2.], and by $\check{H}^i_{zar}(X,\mathcal{F})$ we mean the Čech cohomology as found in [**61**, III.4.]. With $H^i_{\acute{e}t}(X,\bullet)$ and $\check{H}^i_{\acute{e}t}(X,\bullet)$ we mean étale cohomology or Čech étale cohomology, as introduced in [**94**, III.1./III.2.] for the small étale site [**94**, p. 47]. For $Y \to X$ finite étale we have $\check{H}^i_{\acute{e}t}(Y/X,\bullet) := \check{H}^i_{\acute{e}t}(\{Y \to X\},\bullet)$. Here $\bullet$ can mean a quasi-coherent sheaf of $\mathcal{O}_X$-modules ([**94**, II. Corollary 1.6.] – note that we suppress the W), a commutative group scheme ([**94**, III. Corollary 1.7.]), or in special cases certain sheaves of non-abelian groups like $\underline{\mathrm{PGl}}_n$ or $\underline{\mathrm{Gl}}_n$ ([**94**, IV.2.] or [**120**, 2.2.]). In most cases $\bullet$ is a placeholder for a genuine étale sheaf.

Let G be an arbitrary pro-finite group. Denote by $H^i_{grp}(G,M)$ the group cohomology[1] for M a topological G-module as in Shatz [**117**, II.1.]. By $\hat{H}^i_{grp}(G,M)$ we denote the i-th Tate cohomology group for finite G as introduced in [**20**, IV.6. p. 102], which is not to be confused with Čech cohomology. Tate cohomology and group cohomology coincide by definition for $i > 0$ when both are defined, i.e., G is finite. For k a field we have $H^i_{gal}(k,M) := H^i_{grp}(G,M)$, where k^{sep} is the separable closure of k, $G := \mathrm{Gal}(k^{sep}/k)$ the absolute Galois group, a pro-finite group, and M a topological G-module ([**47**], [**117**]). In certain cases non-abelian coefficient groups M are possible ([**47**, Def 2.3.2.]).

We use standard terminology like $\mathbb{G}_m, \mathbb{G}_a, \mu_n, \mathbb{Z}/n\mathbb{Z}, k^{sep}, k^{sep\,*}, \underline{\mathrm{PGl}}_n, \underline{\mathrm{Gl}}_n, \ldots$ for the coefficients. Detailed information on their definition, e.g., Galois action for Galois modules, can be found in the given references.

Notions like the cup product or inflation, restriction, corestriction and transfer, which we use without detailed exposition, can be found in [**101**, I.4. and I.5.], in [**117**, II.2. and II.3.] for group cohomology, and in [**94**, II.3., III. Remark 1.6.(c), III. Theorem 2.20. and V. Proposition 1.20.] for étale cohomology.

We state one of the most central results for different variants of cohomology, namely Hilbert's Theorem 90:

PROPOSITION 3.23 (Hilbert's Theorem 90).
(1) *Let X be a locally noetherian scheme. Then for $n \in \mathbb{N}_0$ arbitrary the natural maps $\check{H}^1_{zar}(X,\underline{\mathrm{Gl}}_n) \to \check{H}^1_{\acute{e}t}(X,\underline{\mathrm{Gl}}_n)$ and $\mathrm{Pic}(X) = H^1_{zar}(X,\mathcal{O}_X^*) \to H^1_{\acute{e}t}(X,\mathbb{G}_m)$ are isomorphisms.*
(2) *Let $U = \mathrm{Spec}(A)$ be an affine scheme, where A is a noetherian local ring. Then for $n \in \mathbb{N}_0$ arbitrary $\check{H}^1_{zar}(U,\underline{\mathrm{Gl}}_n) \cong \check{H}^1_{\acute{e}t}(U,\underline{\mathrm{Gl}}_n) \cong 0$.*
(3) *Let K/k be any separable algebraic field extension. Then for $n \in \mathbb{N}_0$ and $i \in \mathbb{N}$ arbitrary $H^1_{grp}(\mathrm{Gal}(K/k),\underline{\mathrm{Gl}}_n) \cong 0 \cong H^i_{grp}(\mathrm{Gal}(K/k),K)$. In particular $H^1_{gal}(k,k^{sep\,*}) \cong 0 \cong H^i_{gal}(k,k^{sep})$.*

[1]Please note that there is also a similar, but slightly different notion of group cohomology for general groups, which does not involve the topological structure. We do not use it.

(4) *Let K/k be a cyclic field extension with Galois group $\langle\sigma\rangle$. Then* $\forall\alpha \in K^* : N_{K/k}(\alpha) = 1 \Rightarrow \exists\beta \in K^* : \alpha = \beta^{\sigma-\mathrm{id}}$, *and* $\forall\alpha \in K : \mathrm{Tr}_{K/k}(\alpha) = 0 \Rightarrow \exists\beta \in K : \alpha = \beta^\sigma - \beta$.

Proof. The first two variants are most general and are consequences of [**94**, III. Proposition 4.9. and Lemma 4.10.]. The third can be considered a special case of (2) for U the spectrum of a field via the relation between étale and group cohomology discussed below. See also [**101**, VI.2.] and [**47**, Lemma 4.3.7 and 4.3.11]. The last version is the original one, and there are direct computational proofs for it. Viewed from a different angle these statements can be interpreted as vanishing of Tate cohomology $\hat{H}^{-1}_{grp}$ as in [**100**, IV. (3.5) Satz], and thus are a consequence of 2-periodicity in the cyclic case (see [**100**, IV. (3.7) Satz], or more generally [**20**, IV.8. Theorem 5.]) and the third statement. □

Now we collect several standard results that give the link between Azumaya algebras and cohomology groups. In the generality in which we state these results some of them require techniques beyond the scope of this work (e.g., gerbes), and we resort to give mostly references to the proofs. One could simply state the result 3.39, but we like to convey the flavor of this theory without getting too much into the details.

The general assumption in [**94**] is, that schemes are locally noetherian. A careful observation of the proofs there yields that the only statement where local noetherianity is needed is in the second part of 3.38. Also the assumption that every finite subset of the scheme is contained in an open affine subset, which is essentially a projectivity condition (see 3.37), is only needed for this step.

Proposition 3.24 (Noether-Skolem theorem). *Let X be a scheme and $\mathcal{A}$ an X-sAa. The sheaf $\underline{\mathrm{Aut}}(\mathcal{A})$ of automorphisms of $\mathcal{A}$ is étale locally isomorphic to* PGl_n *for some $n \in \mathbb{N}$, where the n may vary with the neighborhood.*

Proof. See [**49**, I, 5.10. & 5.11.]. □

Remark 3.25. This is actually a generalization of the original theorem independently obtained by Noether and Skolem.

Lemma 3.26. *Let X be a scheme. With the obvious morphisms we have a short exact sequence*

$$0 \to \mathbb{G}_m \xrightarrow{diag} \underline{\mathrm{Gl}}_n \to \underline{\mathrm{PGl}}_n \to 0 \tag{3.1.1}$$

both in the category of Zariski sheaves on X and in the category of étale sheaves on X.

Proof. See [**94**, IV. Corollary 2.4.]. □

LEMMA 3.27. *Let X be a scheme. There is an exact sequence of pointed sets which forms the long exact sequence for non-abelian étale sheaves to the sequence* (3.1.1)*:*

$$\begin{gathered} 0 \to \mathbb{G}_m(X) \to \underline{\mathrm{Gl}}_n(X) \to \underline{\mathrm{PGl}}_n(X) \to \\ \check{H}^1_{\acute{e}t}(X, \mathbb{G}_m) \to \check{H}^1_{\acute{e}t}(X, \underline{\mathrm{Gl}}_n) \xrightarrow{q_n} \check{H}^1_{\acute{e}t}(X, \underline{\mathrm{PGl}}_n) \xrightarrow{d'_n} \check{H}^2_{\acute{e}t}(X, \mathbb{G}_m) \end{gathered} \tag{3.1.2}$$

PROOF. See [**94**, III. 4.5. and IV. 2.5.]. □

REMARK 3.28. Note that at least in the special case $X = \mathrm{Spec}(k)$ for k a field $\check{H}^1_{\acute{e}t}(X, \underline{\mathrm{Gl}}_n)$ is not just a pointed set, but also carries an action by the abelian group $\check{H}^1_{\acute{e}t}(X, \mathbb{G}_m)$ (cf. [**57**, p. 10]).

LEMMA 3.29. *Let X be a scheme.*

(1) *The set of isomorphism classes of X-sAa that are of constant rank $n^2 \in \mathbb{N}$ is isomorphic to $\check{H}^1_{\acute{e}t}(X, \underline{\mathrm{Aut}}(\underline{\mathrm{Mat}}(\mathcal{O}_X; n, n))) \cong \check{H}^1_{\acute{e}t}(X, \underline{\mathrm{PGl}}_n)$.*
(2) *The set of isomorphism classes of X-vector bundles that are of constant rank $n \in \mathbb{N}$ is isomorphic to $\check{H}^1_{\acute{e}t}(X, \underline{\mathrm{Aut}}(\mathcal{O}_X^n)) \cong \check{H}^1_{\acute{e}t}(X, \underline{\mathrm{Gl}}_n)$.*
(3) *$q_n(\check{H}^1_{\acute{e}t}(X, \underline{\mathrm{Gl}}_n)) \subset \check{H}^1_{\acute{e}t}(X, \underline{\mathrm{PGl}}_n)$ corresponds exactly to those isomorphism classes of X-sAa whose elements are Brauer equivalent to $\underline{\mathrm{Mat}}(\mathcal{O}_X; n, n)$, which is a representative of the neutral element of the Brauer group.*

PROOF. See [**94**, III. p. 134 and IV. 2.5.]. □

REMARK 3.30. The above results are part of the theory of (twisted) forms and descent theory. There are several references on this subject. For the theory of forms over fields check Serre's [**115**]. A short introduction to cohomology with non-abelian coefficients can be found in Serre's [**116**, VII. Annexe]. For descent theory over rings check [**128**, Part V] of Waterhouse. For the theory over schemes see [**120**, ch 2] next to [**94**].

LEMMA 3.31. *Let X be a scheme, $\mathcal{A}$, resp. $\mathcal{B}$ some X-sAa of globally constant rank $n^2, m^2 \in \mathbb{N}$ with isomorphism class represented by $x \in \check{H}^1_{\acute{e}t}(X, \underline{\mathrm{PGl}}_n), y \in \check{H}^1_{\acute{e}t}(X, \underline{\mathrm{PGl}}_m)$, and let $z \in \check{H}^1_{\acute{e}t}(X, \underline{\mathrm{PGl}}_{nm})$ be associated to $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}$. Then $d'_n(x)d'_m(y) = d'_{nm}(z)$.*
In particular, if $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_X$-modules of globally constant finite rank $m \in \mathbb{N}$ such that $\mathcal{B} \cong \underline{\mathrm{End}}(\mathcal{E})$, then $d'_n(x) = d'_{nm}(z)$. Conversely, if $d'_n(x) = 0 \in \check{H}^2_{\acute{e}t}(X, \mathbb{G}_m)$, then x represents the isomorphism class of some $\underline{\mathrm{End}}(\mathcal{E}')$.

PROOF. First of all $\check{H}^2_{\acute{e}t}(X, \mathbb{G}_m)$ is actually an abelian group, since $\mathbb{G}_m$ is a sheaf of abelian groups. We use exactly this group law in the formulation of the lemma.
The first part is implicit in [**94**, IV. Theorem 2.5. Step 3]. For an explicit account for fields, i.e., $X = \mathrm{Spec}(k)$, consult [**47**, 2.4.]; by Hilbert's Theorem 90 3.23 we have $\check{H}^1_{\acute{e}t}(X, \underline{\mathrm{Gl}}_n) \cong 0$ in this case, and even a group structure on all these pointed sets, which allows to simplify notation drastically.

The second part follows from the first and that by 3.27 we have an exact sequence of pointed sets, i.e., that exactly those elements associated to $\underline{\mathrm{End}}(\mathcal{E})$ get mapped to $0 \in \check{H}^2_{ét}(X, \mathbb{G}_m)$ via d'_{nm} according to 3.29.3. □

LEMMA 3.32. *Let X be a connected scheme. The map $\lambda_X : \dot{\bigcup}_{n\in\mathbb{N}} \check{H}^1_{ét}(X, \underline{\mathrm{PGl}}_n) \to \mathrm{Br}(X)$, given by taking according to 3.29.1 for an $x \in \check{H}^1_{ét}(X, \underline{\mathrm{PGl}}_n)$ a representing X-sAa $\mathcal{A}$ and mapping x to the Brauer equivalence class of $\mathcal{A}$ is well-defined and surjective.*
Furthermore we have for all $n_1, n_2 \in \mathbb{N}$ that $\lambda_X(\mathrm{im}(q_{n_1})) = \{0_{\mathrm{Br}(X)}\}$ and that for $x_1 \in \check{H}^1_{ét}(X, \underline{\mathrm{PGl}}_{n_1}) \subset \dot{\bigcup}_{n\in\mathbb{N}} \check{H}^1_{ét}(X, \underline{\mathrm{PGl}}_n)$ and $x_2 \in \check{H}^1_{ét}(X, \underline{\mathrm{PGl}}_{n_2}) \subset \dot{\bigcup}_{n\in\mathbb{N}} \check{H}^1_{ét}(X, \underline{\mathrm{PGl}}_n)$ with $\lambda_X(x_1) = \lambda_X(x_2)$ we get $d'_{n_1}(x_1) = d'_{n_2}(x_2) \in \check{H}^2_{ét}(X, \mathbb{G}_m)$.

PROOF. From the definition of the Brauer equivalence we see immediately that isomorphic X-sAa are equivalent, hence λ_X is well-defined. On a connected scheme like X a locally free sheaf (of modules) of locally finite rank such as an X-sAa is locally free of globally constant finite rank. Take an arbitrary $[\mathcal{A}] \in \mathrm{Br}(X)$ represented by the X-sAa $\mathcal{A}$ and assume its global rank to be $n^2 \in \mathbb{N}$. By 3.29.1 the isomorphism class of $\mathcal{A}$ is associated to some $x \in \check{H}^1_{ét}(X, \underline{\mathrm{PGl}}_n)$ and thus we have from the definition of λ_X that $\lambda_X(x) = [\mathcal{A}]$, i.e., that λ_X is indeed surjective. By 3.29.3 the claim about $\lambda_X(\mathrm{im}(q_{n_1}))$ is immediate.
Assume now we had x_1, x_2 with the properties sketched in the statement of the lemma. According to the definition of λ_X and of $\mathrm{Br}(X)$ we may represent x_1, x_2 by X-sAa $\mathcal{A}_1, \mathcal{A}_2$ and find locally free sheaves of $\mathcal{O}_X$-modules (of globally constant finite rank) $\mathcal{E}_1, \mathcal{E}_2$ such that

$$\mathcal{A}_1 \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}(\mathcal{E}_1) \cong \mathcal{A}_2 \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}(\mathcal{E}_2).$$

According to 3.31 we get the claimed equality $d'_{n_1}(x_1) = d'_{n_2}(x_2)$. □

PROPOSITION 3.33. *Let X be a connected scheme. Then the map $d''(X) : \mathrm{Br}(X) \to \check{H}^2_{ét}(X, \mathbb{G}_m)$ that to a $[\mathcal{A}] \in \mathrm{Br}(X)$ associates some element of $x \in \lambda_X^{-1}([\mathcal{A}])$ being an isomorphism class of sheaves of globally constant rank $n \in \mathbb{N}$ and sends it to $d'_n(x) \in \check{H}^2_{ét}(X, \mathbb{G}_m)$ is a well-defined group homomorphism.*
It is always injective and maps into the torsion subgroup $\check{H}^2_{ét}(X, \mathbb{G}_m)_{tors}$.

PROOF. By the previous lemma 3.32 $d''(X)$ is a well-defined map. Chasing representing X-sAa through λ_X^{-1} and the d'_n for the corresponding n we get by 3.31 that we have a group homomorphism.
Injectivity follows from the last part of 3.31, which in other words says that preimages of $0 \in \check{H}^2_{ét}(X, \mathbb{G}_m)$ are representing Brauer trivial X-sAa. The final claim about torsion follows from the last part of 3.9, which states that under our condition $\mathrm{Br}(X)$ is torsion and hence clearly its image under any and in particular the considered group homomorphism. □

COROLLARY 3.34. *Let X be a scheme. There is a injective group homomorphism*

$$d''(X) : \mathrm{Br}(X) \to \check{H}^2_{ét}(X, \mathbb{G}_m).$$

Proof. Let C be the set of connected components of X. It is straight forward to see $\mathrm{Br}(X) \cong \prod_{Y\in C} \mathrm{Br}(Y)$ and $\check{H}^2_{\acute{e}t}(X,\mathbb{G}_m) \cong \prod_{Y\in C} \check{H}^2_{\acute{e}t}(Y,\mathbb{G}_m)$. We may apply 3.33 to conclude. ☐

Remark 3.35. Often one finds the abusive notation $\check{H}^1_{\acute{e}t}(X,\underline{\mathrm{PGl}}_\infty)/q(\check{H}^1_{\acute{e}t}(X,\underline{\mathrm{Gl}}_\infty))$, $\prod_{Y\in C} \left(\bigcup_n \check{H}^1_{\acute{e}t}(Y,\underline{\mathrm{PGl}}_n)/q_n(\check{H}^1_{\acute{e}t}(Y,\underline{\mathrm{Gl}}_n))\right)$, $\prod_{Y\in C}\left(\varinjlim_{n|n',\lambda_{n,n'}} \check{H}^1_{\acute{e}t}(Y,\underline{\mathrm{PGl}}_n)/q_n(\check{H}^1_{\acute{e}t}(Y,\underline{\mathrm{Gl}}_n))\right)$ in this context in the literature.

Definition 3.36. A locally noetherian, quasi-compact scheme X such that every finite subset of X is contained in an open affine subset of X is called Čech-suitable.

Remark 3.37. A quasi-projective scheme over an affine noetherian scheme is Čech-suitable. Therefore the most interesting case of quasi-projective k-varieties for k a field is covered by the last result.
For smooth proper schemes over a field Čech-suitable is equivalent to being projective by [**62**, Theorem 9.1.]. See also [**132**] for related notions.

Lemma 3.38. *Let X be a scheme and $\mathcal{F}$ an étale sheaf of abelian groups. Then for $i \in \{0,1\}$:*

$$\phi_i : \check{H}^i_{\acute{e}t}(X,\mathcal{F}) \xrightarrow{\cong} H^i_{\acute{e}t}(X,\mathcal{F}).$$

Let X be Čech-suitable then we get for all $i \in \mathbb{N}_0$:

$$\phi_i : \check{H}^i_{\acute{e}t}(X,\mathcal{F}) \xrightarrow{\cong} H^i_{\acute{e}t}(X,\mathcal{F}).$$

The ϕ_i are functorial.

Proof. See [**94**, III. Theorem 2.17.] and a special case of [**94**, III. Corollary 2.10]. ☐

Proposition 3.39. *Let $H^2_{\acute{e}t}(\ \ ,\mathbb{G}_m), H^2_{\acute{e}t}(\ \ ,\mathbb{G}_m)_{tors} : \mathbf{Sch} \rightsquigarrow \mathbf{Ab}$ denote the usual functors from schemes to abelian groups.* Br *can be considered to be such a functor, too, and there is a natural transformation*

$$d''' : \mathrm{Br} \to H^2_{\acute{e}t}(\ \ ,\mathbb{G}_m),$$

such that for any scheme X the map of abelian groups $d'''(X) : \mathrm{Br}(X) \hookrightarrow H^2_{\acute{e}t}(X,\mathbb{G}_m)$ is injective and, if X has finitely many connected components, it factors via $d(X) : \mathrm{Br}(X) \hookrightarrow H^2_{\acute{e}t}(X,\mathbb{G}_m)_{tors}$.

Proof. The functoriality of Br is proved in [**89**, Cor 42.3.] in a special case which generalizes immediately; the basic idea is that pulling back X-sAa along a morphism $f : Y \to X$ gives a Y-sAa $f^*\mathcal{A}$ and preserves Brauer equivalence.
When X is Čech-suitable the above lemmas give an injective group homomorphism

$$d''' : \mathrm{Br}(X) \xrightarrow{d''(X)} \check{H}^2_{\acute{e}t}(X,\mathbb{G}_m) \xrightarrow{\phi_2\ \cong} H^2_{\acute{e}t}(X,\mathbb{G}_m).$$

Since conditionally $\mathrm{Br}(X)$ is a torsion group by proposition 3.9, this d''' factors through the torsion subgroup of $H^2_{\acute{e}t}(X,\mathbb{G}_m)$, yielding

$$d(X) : \mathrm{Br}(X) \hookrightarrow H^2_{\acute{e}t}(X,\mathbb{G}_m)_{tors}.$$

In general [**48**, V.4.4.] gives for every X an injective homomorphism of abelian groups $d'''(X)$. It remains for any $f : Y \to X$ a morphism of schemes to check $d'''(Y) \circ \mathrm{Br}(f) = H^2_{\acute{e}t}(f,\mathbb{G}_m) \circ d'''(X)$, which is a standard task. To give a reference: [**94**, IV Theorem 2.5.] states that d is canonical (although the proof leaves it to the reader to check it). The statement about torsion follows in the same way as in the special case. □

Definition 3.40. Let X be a scheme. $\mathrm{Br}'(X) := H^2_{\acute{e}t}(X,\mathbb{G}_m)_{tors}$ is called the cohomological Brauer group of X.

To summarize this section: to every scheme one can associate an abelian group, the (Azumaya) Brauer group, which classifies sheaves of Azumaya algebras up to Brauer equivalence, and this assignment is functorial. For schemes having only finitely many connected components there is a natural inclusion of abelian groups from the Brauer group to the torsion subgroup of the second étale cohomology group with coefficients in the group scheme $\mathbb{G}_m$, which in turn is called the cohomological Brauer group.

3.2. Results on the Brauer Group

In this section we start with a comparison of several cohomology theories. Then we discuss spectral sequences and conditions sufficient for the Azumaya Brauer group and the cohomological Brauer group to be isomorphic. We discuss ramification and applications of it to purity and behavior of the Brauer group under morphisms as well as to Brauer groups over number fields. We close by introducing a filtration on the Brauer group and how to use a fibration of a surface to gain insight in its Brauer group. In most situations in this section $\mathrm{Br}(X) = \mathrm{Br}(X)_{tors}$.

3.2.1. Étale Cohomology, Group Cohomology and Singular Cohomology

Definition 3.41. Let X be a scheme and Y an X-scheme. Let G be a finite group that acts by X-automorphisms on Y from the right:

$$a : G \to \mathrm{Mor}_X(Y,Y), \sigma \mapsto (m_\sigma : Y \to Y).$$

We define a morphism of schemes

$$\psi : \coprod_{\sigma\in G} (Y,\sigma) \to Y \times_X Y$$

by requiring $\psi\,|_{(Y,\sigma)} = ((Y,\sigma) \to Y \times_X Y, (y,\sigma) \mapsto (y, m_\sigma(y)))$ for any $\sigma \in G$.
Y is said to be a finite Galois extension (Galois cover) of X with Galois group G and Galois action a, if and only if ψ is an isomorphism of schemes.

REMARK 3.42. In the special case when $X = \mathrm{Spec}(k), Y = \mathrm{Spec}(K)$ for $k \hookrightarrow K$ fields Y/X is finite Galois, if and only if K/k is finite Galois in the usual sense.

LEMMA 3.43. *Let $Y \to X$ be a finite Galois cover with Galois group G and $\mathcal{F}$ an étale sheaf of abelian groups on X. Then for any $i \in \mathbb{N}_0$ there is a natural isomorphism*

$$\check{H}^i_{\acute{e}t}(Y/X, \mathcal{F}) \cong H^i_{grp}(G, \mathcal{F}(Y)),$$

where $\mathcal{F}(Y)$ is an abelian group endowed with a natural G-action from the left.

PROOF. See [**94**, III. Example 2.6.]. □

LEMMA 3.44. *Let k be a field, $G := \mathrm{Gal}(k^{sep}/k)$ its Galois group, M a (discrete) G-module, and $\mathcal{F}$ the associated étale sheaf on $\mathrm{Spec}(k)$. Then for any $i \in \mathbb{N}_0$ there are natural isomorphisms*

$$H^i_{\acute{e}t}(\mathrm{Spec}(k), \mathcal{F}) \cong H^i_{grp}(G, M) = H^i_{gal}(k, M) \cong \varinjlim_{K/k \text{ finite Galois}} H_{grp}(Gal(K/k), M^{\mathrm{Gal}(k^{sep}/K)}).$$

PROOF. See [**94**, II.1.9./11. and III. Example 1.7.] and [**117**, II.2. Theorem 7. and Corollary 1.]. □

REMARK 3.45. For $i \in \{0, 1\}$ the analogous statements of the last two lemmas for (étale sheaves of) nonabelian groups are also true.

DEFINITION 3.46. Let X be a connected scheme and x a geometric point, i.e., a morphism $\mathrm{Spec}(k^{alg}) \to X$, where k^{alg} is an algebraically closed field. Let $\mathbf{F\acute{E}tC_X}$ be the full subcategory of finite étale connected schemes over X of the usual category of X-schemes. Define a so called fiber functor $F : \mathbf{F\acute{E}tC_X} \rightsquigarrow \mathbf{Set}$ by $F_x(Y) = \mathrm{Hom}_{\mathbf{F\acute{E}tC_X}}(x, Y) = \mathrm{Hom}_X(x, Y)$. The set of automorphisms of a functor is the set of natural transformations $F_x \to F_x$ admitting a two-sided inverse natural transformation. Define the étale fundamental group with base point in x to be this group (see [**123**, Definition 5.4.1])

$$\pi_1^{\acute{e}t}(X, x) := \mathrm{Aut}(F_x).$$

REMARK 3.47. Since $\pi_1^{\acute{e}t}$ only depends on the choice of the base point x up to an isomorphism canonical up to inner isomorphisms (see [**94**, I. Remark 5.1. (a)] where a slightly different but equivalent definition of $\pi_1^{\acute{e}t}$ is given), we allow ourselves to write $\pi_1^{\acute{e}t}(X)$ and ignore the choice of a base point.

PROPOSITION 3.48. *Let X be a smooth projective variety over an algebraically closed field k endowed with a fixed embedding $k \hookrightarrow \mathbb{C}$. Let X^{an} be the associated compact complex manifold as in remark 2.43. Denote by $\widehat{\pi_1^{an}(X^{an})}$ the topological group completion of the analytic fundamental group of X^{an} with respect to the topology induced by the system of subgroups of finite index. Then*

$$\pi_1^{\acute{e}t}(X) \cong \widehat{\pi_1^{an}(X^{an})}.$$

PROOF. See [**94**, I. Remark 5.1.(c)] for the case $k = \mathbb{C}$. By pro-representability of a fiber functor as in definition 3.46 via Galois covers (see [**94**, I.5.] or [**123**, Proposition 5.4.6]) it suffices to show a correspondence of certain Galois covers. For arbitrary k we can lift a connected Galois cover by base change to a corresponding one over $\mathbb{C}$. Any connected Galois cover over $\mathbb{C}$ for a variety with field of definition k corresponds Zariski locally to extensions of sheaves of rings which are finite and thus do not involve adjoining additional transcendental elements, i.e., the cover can be defined over k. Thus $\pi_1^{\acute{e}t}(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\mathbb{C})) \cong \pi_1^{\acute{e}t}(X)$. Cf. [**95**, Remark 21.10]. □

PROPOSITION 3.49. *Let X be a smooth projective variety over an algebraically closed field k endowed with a fixed embedding $k \hookrightarrow \mathbb{C}$. Let $\underline{M}$ be an étale sheaf of finite abelian groups on X and M the induced local system on X^{an}. Then for $i \in \mathbb{N}_0$*

$$H^i_{sing}(X^{an}, M) \cong H^i_{\acute{e}t}(X, \underline{M}).$$

PROOF. See [**94**, III. Theorem 3.12. and Lemma 3.15.] for $k = \mathbb{C}$, and then apply proper base change, e.g., see [**95**, Corollary 17.8. (b)]. □

PROPOSITION 3.50. *Let X be a smooth quasi-projective curve over a field k endowed with a fixed embedding $k \hookrightarrow \mathbb{C}$. Let M be the stalk at the generic point associated to an étale sheaf of finite abelian groups $\underline{M}$ as a $\pi_1^{\acute{e}t}(X)$-module. Then for $i \in \mathbb{N}_0$:*

$$H^i_{grp}(\pi_1^{\acute{e}t}(X), M) \cong H^i_{\acute{e}t}(X, \underline{M}).$$

In particular for $i \in \mathbb{N}_{>1}$ we have $H^i_{grp}(\pi_1^{\acute{e}t}(X), M) \cong 0$.

PROOF. See [**94**, V. Lemma 2.17.] and the previous proposition. The tame étale fundamental group for fields of characteristic 0 is the whole étale fundamental group ([**94**, V. Proposition 1.8.]), and sheaves of finite abelian groups are constructible ([**94**, I. Example 5.2.(e)]). □

COROLLARY 3.51. *Let X be a smooth quasi-projective curve over an algebraically closed field k endowed with a fixed embedding $k \hookrightarrow \mathbb{C}$. Let M be the stalk at the generic point associated to an étale sheaf of finite abelian groups $\underline{M}$ as a $\pi_1^{\acute{e}t}(X)$-module. Then $H^1_{grp}(\widehat{\pi_1^{an}(X^{an})}, M) \cong H^1_{grp}(\pi_1^{\acute{e}t}(X), M) \cong H^1_{\acute{e}t}(X, \underline{M})$.*

PROOF. Combine the previous two propositions. □

REMARK 3.52. In applications to varieties over number fields there is a finite field extension for the base field that trivializes the sheaf of modules $\underline{M}$, i.e., the pullback to the extension is isomorphic to some $\bigoplus_{i\in\{1,2,\ldots,m\}} \mathbb{Z}/n_i\mathbb{Z}$ as étale sheaves, and the topological fundamental group can act nontrivially on it. But for locally finite sheaves of modules the action factors through a finite quotient of the fundamental group, thus is trivialized after a further finite field extension. This may be carried out effectively as well as the associated cohomology computations.

3.2.2. Certain Spectral Sequences

An introduction to spectral sequences can be found in many books on homological algebra, e.g., [**109**] of Rotman, who discusses them in the context of (co-)homology for categories of modules over (not necessarily commutative) rings. In [**117**] one finds a concise description giving explicit formula, and [**94**] contains a short appendix summarizing results concerning them. A standard reference is also Grothendieck's Tohoku paper [**54**].

One should be careful in distinguishing the definition of a spectral sequence over a given abelian category from the many constructions giving rise to them in various settings. The abelian categories involved are usually the category of abelian groups, module categories over a ring, or sheaf versions of them. The category of (topological) G-modules for G a (topological or pro-finite) group is essentially a special case. We assume that the reader is familiar with spectral sequences as covered and used in these sources.

THEOREM 3.53 (Grothendieck spectral sequence). *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be additive functors between abelian categories with enough injective objects. Assume that G is left exact, and $F(E) \in \mathcal{B}$ is right G-acyclic for all injective objects $E \in \mathcal{A}$. Then for $A \in \mathcal{A}$, there is a convergent spectral sequence functorial in A:*

$$E_2^{p,q} = ((R^pG)(R^pF))A \Rightarrow (R^n(GF))A.$$

PROOF. See [**109**, Theorem 10.47], which does not mention functoriality, and [**50**, III.7. Theorem 7] for functoriality. □

THEOREM 3.54 (Leray spectral sequence). *Let $\pi : Y \to X$ be a morphism of schemes and $\mathcal{F}$ an étale sheaf of abelian groups on Y. Then there is a convergent spectral sequence functorial in $\mathcal{F}$:*

$$H^p_{\acute{e}t}(X, R^q\pi_*\mathcal{F}) \Rightarrow H^n_{\acute{e}t}(Y, \mathcal{F}).$$

PROOF. This is a special case of the Grothendieck spectral sequence 3.53. One mechanically checks all the conditions. We may also derive it as a special case of [**94**, III. Theorem 1.18.] (π induces a continuous morphism between the étale sites of X and Y by [**94**, II. Example 2.1.(c)]), or as a special case of [**124**, II.(1.4.3.)] (where in the notation of Tamme $Y' = Y$ is the trivial étale cover). □

THEOREM 3.55 (Hochschild-Serre spectral sequence). *Let $\pi : Y \to X$ be a finite Galois cover with Galois group G' and $\mathcal{F}$ an étale sheaf of abelian groups on X. Then there is a convergent spectral sequence functorial in $\mathcal{F}$:*

$$H^p_{grp}(G', H^q_{\acute{e}t}(Y, \pi^*\mathcal{F})) \Rightarrow H^n_{\acute{e}t}(X, \mathcal{F}).$$

PROOF. This is a special case of the Grothendieck spectral sequence 3.53, where $\mathcal{A} = \mathbf{Sh}_{X,\acute{e}t}$ is the category of étale sheaves of abelian groups on X, $\mathcal{B} = \mathbf{Mod}_{G'}$ is the category of G'-modules, and $\mathcal{C} = \mathbf{Ab}$ is the category of abelian groups. The functor $F(\mathcal{F}') := \mathcal{F}'(Y) = \Gamma_Y(\pi^*\mathcal{F}')$ is evaluating a sheaf on X at Y,

which gives at first an abelian group, but the G'-action on $Y \to X$ from the right gives rise to a G'-action on $A' := \mathcal{F}'(Y)$ from the left. $G(A') := A'^{G'}$ is taking the G'-invariants of a G'-module.
Using the sheaf property in the special case of a Galois cover (see [**94**, II. Proposition 1.4.]), we get $(\Gamma_Y(\pi^*\mathcal{F}))^{G'} = (G \circ F)(\mathcal{F}) \cong \Gamma_X(\mathcal{F})$.
By definition, e.g., [**94**, III. Definition 1.5.(a/b)], the right derived functor of the section functors F and $G \circ F$ are étale cohomology functors, and the right derived functors of taking invariants of a group are the group cohomology functors. See [**50**, III.7. Exercise 1.] for an explicit statement in the literature. Thus in this case the Grothendieck spectral sequence takes on the indicated form. □

REMARK 3.56. $\pi^*\mathcal{F}$ is the restriction of an étale sheaf to an étale cover. For étale covers, π^* is an exact functor by [**94**, II. Proposition 2.6. and II.3. p.68].

The following statement could be generalized to arbitrary infinite Galois extensions of schemes. Since we only need a special case, we avoid the technicalities in defining arbitrary Galois covers and associated pro-Galois systems. The interested reader is referred to [**94**, I.5. and III. Remark 2.21.(b)].

PROPOSITION 3.57. *Let k be a field, X be a quasi-projective k-scheme and $\mathcal{F}$ an étale sheaf of abelian groups on X. Let $X^{sep} \xrightarrow{\pi} X$ be the canonical morphism. Then there is a convergent spectral sequence functorial in $\mathcal{F}$:*

$$H^p_{gal}(k, H^q_{ét}(X^{sep}, \pi^*\mathcal{F})) \Rightarrow H^n_{ét}(X, \mathcal{F}).$$

PROOF. We carry out the standard arguments:
Since X is quasi-projective over k, it can be embedded into some $\mathbb{P}^n_k$ and is therefore noetherian. By [**61**, II. Theorem 4.9.] X is of finite type and separated. By [**61**, II. Exercise 3.3.(a)] finite type implies quasi-compact. Since $X \to \mathrm{Spec}(k)$ is separated, we have that the diagonal morphism $X \to X \times_k X$ is a closed embedding. Being closed easily implies finite type and thus yields that the diagonal morphism is quasi-compact, so by definition X is quasi-separated.
Let $(k_i)_{i\in I}$ be the filtered injective system of finite Galois subextensions of k^{sep}/k giving rise to a filtered projective system $(X_i)_{i\in I} = (X_{k_i})_{i\in I}$. Any inclusion $k_i \hookrightarrow k_j$ of the system is finite, and so by [**94**, I. Proposition 1.3.(c)] the associated $X_j \to X_i$ is finite and hence affine. As a special case $X_i \xrightarrow{\pi_i} X$ is finite, and by [**94**, I. Proposition 1.4.] is thus of finite type and separated. The later two properties are preserved by composition of morphisms of the respective property ([**61**, II. Exercise 3.13.(c) and II. Corollary 4.6.(b)]) and hence by the above arguments the X_i are quasi-compact and quasi-separated.
We have now checked all the conditions to apply [**124**, II.(6.3)] or [**94**, III. Lemma 1.16.], which state that $X^{sep} \cong \varprojlim X_i$ and

$$\varinjlim H^p_{ét}(X_i, \pi_i^*\mathcal{F}) \cong H^p_{ét}(X^{sep}, \pi^*\mathcal{F}).$$

Next we apply 3.55 to all the $\pi_i : X_i \to X$ and the associated data simultaneously and get a directed system of spectral sequences

$$H^p_{grp}(Gal(k_i/k), H^q_{\acute{e}t}(X_i, \pi_i^*\mathcal{F})) \Rightarrow H^n_{\acute{e}t}(X, \mathcal{F}),$$

which are compatible; however, we refrain from explicitly showing compatibility due to its lengthy technicalities.
Now group cohomology is compatible with directed limits ([**117**, II.2. Theorem 7.]), as is étale cohomology in this case by the above discussion, so we arrive at

$$H^p_{grp}(Gal(k^{sep}/k), H^q_{\acute{e}t}(X^{sep}, \pi^*\mathcal{F})) \Rightarrow H^n_{\acute{e}t}(X, \mathcal{F})$$

which by definition of Galois cohomology was claimed.
Functoriality follows from the functoriality of direct limits. □

PROPOSITION 3.58. *Let k be a field, X a quasi-projective k-scheme and $\mathcal{F}$ an étale sheaf of abelian groups on X. Let either $K := k^{sep}$ or K a finite Galois extension of k, and denote $G := \mathrm{Gal}(K/k)$ and $Y := X_K \xrightarrow{\pi} X$. Then we have a (functorial) exact sequence*

$$\begin{aligned} 0 \to &H^1_{grp}(G, H^0_{\acute{e}t}(Y, \pi^*\mathcal{F})) \to H^1_{\acute{e}t}(X, \mathcal{F}) \to H^0_{grp}(G, H^1_{\acute{e}t}(Y, \pi^*\mathcal{F})) \to \\ &H^2_{grp}(G, H^0_{\acute{e}t}(Y, \pi^*\mathcal{F})) \to \ker(H^2_{\acute{e}t}(X, \mathcal{F}) \to H^0_{grp}(G, H^2_{\acute{e}t}(Y, \pi^*\mathcal{F}))) \to \\ &H^1_{grp}(G, H^1_{\acute{e}t}(Y, \pi^*\mathcal{F})) \to H^3_{grp}(G, H^0_{\acute{e}t}(Y, \pi^*\mathcal{F})). \end{aligned}$$

PROOF. This is a special case of the exact sequence in low degree terms for first quadrant spectral sequences as in the setup of 3.57 and 3.55. The well-known 5-term sequence (see 3.60) is the basic version of it, but does not suffice in our case. An account for the given sequence can be found in [**94**, Appendix B, p.309]. □

COROLLARY 3.59. *There is the analogous exact sequence for the setting of 3.54:*

$$\begin{aligned} 0 \to &H^1_{\acute{e}t}(X, R^0\pi_*\mathcal{F}) \to H^1_{\acute{e}t}(X, \mathcal{F}) \to H^0_{\acute{e}t}(X, R^1\pi_*\mathcal{F}) \to \\ &H^2_{\acute{e}t}(X, R^0\pi_*\mathcal{F}) \to \ker(H^2_{\acute{e}t}(X, \mathcal{F}) \to H^0_{\acute{e}t}(X, R^2\pi_*\mathcal{F})) \to \\ &H^1_{\acute{e}t}(X, R^1\pi_*\mathcal{F}) \to H^3_{\acute{e}t}(X, R^0\pi_*\mathcal{F}) \end{aligned}$$

We apply the Lyndon-Hochschild-Serre spectral sequence in section 4.3 and in appendix A. The focus in the appendix, however, is on explicit formulas for the resulting exact sequence in low degree terms. There are different versions of this theorem. We state it for profinite groups with standard topology. In the application for finite groups all topological requirements are trivially met, hence they can be ignored.

PROPOSITION 3.60 (Lyndon-Hochschild-Serre spectral sequence). *Let G be a profinite group and*

$$0 \to N \xrightarrow{\pi} G \to Q \to 0$$

a short exact sequence of groups with N a closed normal subgroup. Let A be a G-module (with discrete topology). Then there is a convergent spectral sequence functorial in A:

$$H^p_{grp}(Q, H^q_{grp}(N, \pi^*A)) \Rightarrow H^n_{\acute{e}t}(G, A).$$

Furthermore the functorial exact sequence in low degree terms up to the 5*-th term (=* 5*-term sequence, in this case also called inflation-restriction sequence) associated to this spectral sequence is*

$$0 \to H^1_{grp}(Q, (\pi^* A)^N) \xrightarrow{\inf} H^1_{grp}(G, A) \xrightarrow{\text{res}} (H^1_{grp}(N, \pi^* A))^Q \to H^2_{grp}(Q, (\pi^* A)^N) \xrightarrow{\inf_2} H^2_{grp}(G, A)$$

Proof. This result is analogous to the one above and proved similarly. By definition $H^0_{grp}(G, A) = A^G$. See [**117**, p. 51]. □

3.2.3. On Isomorphy of Azumaya and Cohomological Brauer Group

Theorem 3.61 (A result of Gabber). *Let X be a quasi-compact separated scheme endowed with an ample invertible sheaf $\mathcal{L}$ in the sense of* [**55**, II. Définition (4.5.3)]. *Then the natural transformation d of 3.39 gives an isomorphism*

$$\mathrm{Br}(X) \xrightarrow{d(X)\ \cong} \mathrm{Br}'(X).$$

Proof. See [**30**]. □

Corollary 3.62. *Let k be a field and X a quasi-projective k-scheme. Then* $\mathrm{Br}(X) \xrightarrow{d(X)\ \cong} \mathrm{Br}'(X)$.

Proof. Since X is quasi-projective as a k-scheme, there is an embedding $X \hookrightarrow \mathbb{P}^n_k$ giving rise to an ample invertible sheaf of hyperplane sections. Quasi-projective implies quasi-compact and separated. Then we apply 3.61. □

Proposition 3.63. *If X is a regular noetherian integral scheme with residue field of the generic point denoted by $k(X)$, then $H^2_{ét}(X, \mathbb{G}_m) \hookrightarrow H^2_{ét}(\mathrm{Spec}(k(X)), \mathbb{G}_m)$ and $\mathrm{Br}'(X) = H^2_{ét}(X, \mathbb{G}_m)$.*

Proof. The inclusion is proved in [**94**, III. Example 2.22.]. The main idea is working with several exact sequences and showing vanishing for some modules of that sequences. We state the ingredients.
For single points, i.e., spectra of fields, there is a short exact sequence of constant sheaves

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{Q}} \to \underline{\mathbb{Q}/\mathbb{Z}} \to 0.$$

For quasi-compact (e.g., noetherian) regular integral schemes, there is a short exact sequence of étale sheaves of abelian groups on X

$$0 \to \mathbb{G}_{m,X} \to \iota_* \mathbb{G}_{m,\mathrm{Spec}(k(X))} \to D_X \to 0,$$

where $D_X = \bigoplus_{x \in X_1} \iota_{x\,*} \underline{\mathbb{Z}}$ is the sheaf of Weil devisors on X with X_1 the set of codimension 1 points of X and $\iota : \mathrm{Spec}(k(X)) \hookrightarrow X$ and $\iota_x : x \hookrightarrow X$ are the inclusions for the generic point respectively the point x by [**94**, II. Example 3.9.]. By [**117**, II.3. Corollary 2, p.35] group cohomology of pro-finite groups such as Galois groups are torsion, and torsion is annihilated by uniquely divisible groups like $\mathbb{Q}$. Thus for $x \in X_1$ and $r \in \mathbb{N}$

$$H^r_{ét}(x, \underline{\mathbb{Q}}) \cong H^r_{grp}(\mathrm{Gal}(\kappa(x)^{sep}/\kappa(x)), \mathbb{Q}) \cong 0.$$

By Hilbert's Theorem 90 3.23 we have

$$H^1_{ét}(x, \mathbb{G}_m) \cong H^1_{grp}(\mathrm{Gal}(\kappa(x)^{sep}/\kappa(x)), \kappa(x)^{sep\ *}) \cong 0.$$

We combine the first vanishing, the long exact sequence in cohomology for the first sequence and the associated exact sequence in low degree terms of the Leray spectral sequence 3.54 for $\pi = \iota_x$ and $\underline{\mathbb{Z}}$, leading to a result about cohomology of D_X since cohomology is compatible with direct sums.
Finally we combine this result on D_X, the vanishing result of Hilbert's Theorem 90, the long exact sequence in cohomology for the second sequence and the Leray spectral sequence 3.54 for $\pi = \iota$ and $\mathbb{G}_{m,\mathrm{Spec}(k(X))}$, yielding the desired inclusion.
By 3.44 $H^2_{ét}(\mathrm{Spec}(k(X)), \mathbb{G}_m) \cong H_{grp}(\mathrm{Gal}(k(x)), k(x)^{sep\ *})$ holds. Now a subgroup of a torsion group is itself a torsion group and thus coincides with its torsion part, yielding the stated equality. □

Remark 3.64. Regular is important for the isomorphism between groups of Weil divisors and Cartier devisors; see [**61**, II Proposition 6.11.1A], which additionally assumes separatedness due to its definition of Weil devisors but that is of no relevance to us. The second short exact sequence above with the sheaf of Cartier divisors replacing the sheaf of Weil divisors is essentially the definition of Cartier divisors. One could weaken regular to locally factorial.

Corollary 3.65. *If X is a quasi-projective regular k-variety then* $\mathrm{Br}(X) \xrightarrow{d(X)\ \cong} \mathrm{Br}'(X) = H^2_{ét}(X, \mathbb{G}_m)$ *is a natural isomorphism.*

Proof. Since quasi-projective k-varieties are noetherian and integral, we can combine 3.63 and 3.62. Remember that d was natural by proposition 3.39. □

Remark 3.66. Grothendieck in [**49**, II, Remarques 1.11.b)] describes how a non-smooth but normal $\mathbb{C}$-surface Y constructed by Mumford in [**98**, p.16] by blowing up 15 points in a certain configuration and contracting an exceptional curve on the resulting surface has a nontrivial free quotient for $H^2_{ét}(Y, \mathbb{G}_m)$, thus, $\mathrm{Br}'(Y) \ncong H^2_{ét}(Y, \mathbb{G}_m)$.
In [**36**, Corollary 3.11] Edidin, Hassett, Kresch and Vistoli give a simple example of a non-separated $\mathbb{C}$-scheme Z glued from two non-smooth $\mathbb{C}$-surfaces such that $\mathrm{Br}(Z) \xrightarrow{d(Z)} \mathrm{Br}'(Z)$ is not surjective, and $\mathrm{Br}(Z)$ and $\mathrm{Br}'(Z)$ are not even isomorphic by any other morphism.

Proposition 3.67. *Let R be a Henselian (and therefore local) ring with residue field k. Then* $\mathrm{Br}(k) \cong \mathrm{Br}(\mathrm{Spec}(R)) \xrightarrow{d(\mathrm{Spec}(R))\ \cong} \mathrm{Br}'(\mathrm{Spec}(R)) = H^2_{ét}(\mathrm{Spec}(R), \mathbb{G}_m)$.

Proof. See [**94**, IV. Corollary 2.12./13.], and note that the cohomological Brauer group in [**94**] is defined without taking the torsion part. □

3.2.4. The Residue Map and the Unramified Brauer Group

References for the notion of ramification of Azumaya algebras over rings are the lecture notes of Saltman [**111**, Ch. 9/10], or the book of Pierce [**103**, Ch. 17]. The paper of Colliot-Thélène and Swinnerton-Dyer [**28**, §1] gives a summary on residue maps in the context of arithmetic geometry. We resort to stating the results.

PROPOSITION 3.68.

(1) *(Auslander-Goldman) Let R be a noetherian regular domain with fraction field K, i.e., there is a morphism $f : \mathrm{Spec}(K) \hookrightarrow \mathrm{Spec}(R)$. Then $f^* : \mathrm{Br}(R) \to \mathrm{Br}(K)$ is injective.*

(2) *(Hoobler) Let R be a noetherian regular domain and $\mathcal{R}_1$ the set of height 1 prime ideals. Then $\mathrm{Br}(R) \cong \bigcap_{P \in \mathcal{R}_1} \mathrm{Br}(R_P)$.*

(3) *(Auslander-Brumer) Let R be a discrete valuation ring with fraction field K, residue field k, and residue characteristic p. Let $\blacksquare\{p\}$ denote the prime to p part of an abelian group. One has an exact sequence*

$$0 \to \mathrm{Br}(R)\{p\} \to \mathrm{Br}(K)\{p\} \xrightarrow{\partial'_R} \mathbf{X}(\mathrm{Gal}(k^{sep}/k))\{p\} \to 0,$$

where we denoted the character group of $\mathrm{Gal}(k^{sep}/k)$ by $\mathbf{X}(\mathrm{Gal}(k^{sep}/k)) := \mathrm{Hom}_{cont}(\mathrm{Gal}(k^{sep}/k), \mathbb{Q}/\mathbb{Z}) \cong H^1_{grp}(\mathrm{Gal}(k^{sep}/k), \mathbb{Q}/\mathbb{Z}) \cong H^2_{grp}(\mathrm{Gal}(k^{sep}/k), \mathbb{Z})$. This character group is closely related to the Pontrayagin dual.

(4) *(Auslander-Brumer, variant) Let R be a discrete valuation ring with fraction field K and perfect residue field k. There is an exact sequence*

$$0 \to H^2_{\acute{e}t}(\mathrm{Spec}(R), \mathbb{G}_m) \to H^2_{\acute{e}t}(\mathrm{Spec}(K), \mathbb{G}_m) \xrightarrow{\partial_R} \mathbf{X}(\mathrm{Gal}(k^{sep}/k)).$$

If R is Henselian (e.g., complete), then ∂_R is surjective, and we have a short exact sequence:

$$0 \to \mathrm{Br}(R) \to \mathrm{Br}(K) \xrightarrow{\partial_R} \mathbf{X}(\mathrm{Gal}(k^{sep}/k)) \to 0.$$

PROOF. In the cited references a regular ring is noetherian by definition.
For the first part see [**111**, Theorem 9.6.] or [**7**, Theorem 7.2.].
For the second part see [**111**, Theorem 9.7.] or [**68**, Main Theorem]. By abuse of notation we regarded all Brauer groups $\mathrm{Br}(R_P)$ as subgroups of $\mathrm{Br}(K)$ where K is the fraction field of R using the inclusion of the first part.
For the third part see [**111**, Theorem 10.3.] or [**6**].
For the forth part see [**49**, III, Proposition 2.1.] and [**94**, III. Example 2.22.(c)], which are stated for $H^2_{\acute{e}t}(\bullet, \mathbb{G}_m)$ instead of $\mathrm{Br}(\bullet)$, which yields the claim by proposition 3.67. The techniques used in the references are outlined in the proof of proposition 3.63. □

REMARK 3.69. The generalized Noether-Skolem theorem, albeit different from the above, is also referred to as Auslander-Goldman theorem (see, e.g., [**49**, I, Théorèm 5.10.]).

Definition 3.70. The maps ∂'_R and ∂_R above are called residue maps (ramification maps).

Remark 3.71. In applications we replace domain and codomain of the residue map by isomorphic groups and still call it the residue map. The invariant map inv for local fields of [**28**, §1] is an example.

Remark 3.72. The residue map is natural in that it forms commutative diagrams with restriction and corestriction functors for certain extension of discrete valuation rings $R \subset S$. See [**28**, §1], [**116**, XII.3. Exercise 2.], (cf. [**46**, Proposition 8.2.]) or [**111**, Theorem 10.4.]. Explicitly let L/K be a finite extension of discretely valued fields with perfect residue fields l and k. Let e be the ramification index and m_e the multiplication by e map. Let R_K be a discrete valuation ring with field of fractions K' with perfect residue field k'. Let L'/K' be a finite field extension and let R_L be the integral closure of R_K in L, which is a semilocal ring with residue fields $(l'_i)_{i\in I}$ and associated residue maps $(\partial_i)_{i\in I}$. Then the following diagrams commute

$$\begin{tikzcd} \mathrm{Br}(L) \arrow[r, "\partial_L"] & \mathbf{X}(\mathrm{Gal}(l^{alg}/l)) \\ \mathrm{Br}(K) \arrow[u, "\mathrm{res}_{L/K}"] \arrow[r, "\partial_K"] \arrow[ur, phantom, "\circlearrowleft"] & \mathbf{X}(\mathrm{Gal}(k^{alg}/k)) \arrow[u, "m_e \circ \mathrm{res}_{l/k}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Br}(L') \arrow[r, "\sum_{i\in I}\partial_i"] \arrow[d, "\mathrm{cores}_{L'/K'}"'] \arrow[dr, phantom, "\circlearrowleft"] & \bigoplus_{i\in I} \mathbf{X}(\mathrm{Gal}(l_i'^{alg}/l'_i)). \arrow[d, "\sum_{i\in I}\mathrm{cores}_{l_i/k}"] \\ \mathrm{Br}(K') \arrow[r, "\partial_{K'}"] & \mathbf{X}(\mathrm{Gal}(k'^{alg}/k')) \end{tikzcd}$$

Definition 3.73. Let k be a field and K/k any field extension. Let $\mathcal{R}_{dvr,reg}(K/k)$ be the set of all discrete valuation rings R with $k \subset R \subset K$ that are regular domains. By proposition 3.68(1) we may identify $\mathrm{Br}(R)$ for $R \in \mathcal{R}_{dvr,reg}(K/k)$ with its image in $\mathrm{Br}(K)$ and thus define $\bigcap_{R\in\mathcal{R}_{dvr,reg}(K/k)} \mathrm{Br}(R) =: \mathrm{Br}_{un}(K/k) =: \mathrm{Br}_{un}(K) \subset \mathrm{Br}(K)$ to be the unramified Brauer group of K (over k).

Proposition 3.74. *The unramified Brauer group is functorial, i.e., if k is a field, then Br_{un} can be extended to a functor from the category of fields over k to abelian groups.*

Proof. See [**111**, Proposition 10.5. a)]. The proof is straightforward relying on the behavior of discrete valuation subrings of a field with respect to field extensions. □

Proposition 3.75. *Let k be a field and X a projective regular k-variety with function field $k(X)$. Then $\mathrm{Br}(X) \cong \mathrm{Br}_{un}(k(X)/k)$.*

Proof. This is [**111**, Proposition 10.5. c)]; due to its brevity we repeat it here.
Look at the functor $\underline{Br}_X : \mathbf{Top}_{zar}(X) \to \mathbf{Ab}$ from the categorical Zariski topology, i.e., the category of Zariski open subsets of X with open inclusions as morphisms, to the category of abelian groups that is defined on objects via $U \mapsto \mathrm{Br}(U)$. Since X is regular, this contravariant functor is a sheaf by [**94**, IV. Remark 2.10.]. We

may choose an affine open cover $X = \bigcup_{i \in I} \mathrm{Spec}(R_i)$ with the R_i regular noetherian domains. By the sheaf property of $\underline{Br}_X$ and by proposition 3.68(2), we get $\mathrm{Br}(X) = \bigcap_{i \in I} \bigcap_{P \in \mathcal{R}_{i,1}} \mathrm{Br}(R_{i,P})$. By regularity of the R_i all $R_{i,P}$ are regular, thus for all $i \in I$ we have $\mathcal{R}_{i,1} \subset \mathcal{R}_{dvr,reg}(K/k)$ and therefore $\mathrm{Br}_{un}(k(X)/k) \subset \mathrm{Br}(X)$. Let $R \in \mathcal{R}_{dvr,reg}(K/k)$ be arbitrary. By the valuative criterion of properness (see [**61**, II. Theorem 4.7]), and since projective varieties are proper (see [**61**, II. Theorem 4.9]), we get a morphism $\mathrm{Spec}(R) \to X$, where the closed point of $\mathrm{Spec}(R)$ is mapped into the closure of some codimension 1 point P. By functoriality of Br, we get inclusions $\mathrm{Br}(X) \subset \mathrm{Br}(\mathcal{O}_{X,P}) \subset \mathrm{Br}(\mathrm{Spec}(R))$. Since R was arbitrary, we get $\mathrm{Br}(X) \subset \mathrm{Br}_{un}(k(X)/k)$. □

Remark 3.76. We may weaken projective to proper as is easily seen from the proof.

3.2.5. Birational Invariance and Purity of the Brauer Group

In this subsection we assume all schemes to have finitely many connected components when talking about birational morphisms.

Lemma 3.77. *Let k be a field and X a quasi-projective k-scheme. Then X is excellent.*

Proof. By [**55**, IV. Scholie (7.8.3) (iii)] any field is an excellent ring, i.e, $\mathrm{Spec}(k)$ is an excellent scheme. By [**55**, IV. Proposition (7.8.6) (i)], any scheme locally of finite type over an excellent scheme is excellent, and since quasi-projective k-schemes are as subschemes of a certain $\mathbb{P}^n_k$ of finite type over $\mathrm{Spec}(k)$, we have X is excellent. □

Proposition 3.78.
(1) *Let $f : X \to Y$ be a proper birational morphism of regular noetherian schemes of dimension ≤ 2. Then $f^* = \mathrm{Br}(f) : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is an isomorphism.*
(2) *Let $f : X \to Y$ be a proper birational morphism of regular excellent schemes of characteristic 0. Then $\tilde{f}^* = H^2_{ét}(f, \mathbb{G}_m) : H^2_{ét}(Y, \mathbb{G}_m) \to H^2_{ét}(X, \mathbb{G}_m)$ is an isomorphism.*
(3) *Let k be a field and $f : X \to Y$ be a birational morphism of regular noetherian proper k-schemes.*
   (a) *If $\dim(X) = \dim(Y) \leq 2$, then the three morphisms $f^* : \mathrm{Br}(Y) \to \mathrm{Br}(X), d(\bullet) : \mathrm{Br}(\bullet) \hookrightarrow \mathrm{Br}'(\bullet) = H^2_{ét}(\bullet, \mathbb{G}_m)$, where $\bullet \in \{X, Y\}$ are isomorphisms.*
   (b) *If $p \neq \mathrm{char}(k)$ is a prime and $\blacksquare[p]$ signifies the p-power torsion part of an abelian group, then the morphism $\tilde{f}^*[p] : H^2_{ét}(Y, \mathbb{G}_m)[p] \to H^2_{ét}(X, \mathbb{G}_m)[p]$ is an isomorphism.*

PROOF. The first statement is [**49**, III, Corollaire 7.2.], the second is [**49**, III.7.3.], where the obviously necessary condition of f being proper[2] is missing as indicated in a footnote, and the last statement is [**49**, III.7.5.], making explicit what it means in this context to be a birational invariant. $\square$

REMARK 3.79. The condition on the characteristic in the second part comes from resolution of singularities on which the proof relies, which was in the needed generality only proved in characteristic 0 by Hironaka, see proposition 2.32.

PROPOSITION 3.80. *Let k be a field, and let $f : X \to Y$ be a birational morphism of smooth projective k-varieties. Then $f^* = \mathrm{Br}(f) : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is an isomorphism.*

PROOF. We give two proofs, one for $\mathrm{char}(k) = 0$ following Grothendieck in [**49**], and the other for arbitrary k as indicated in [**127**] and [**111**].
Assume $\mathrm{char}(k) = 0$. By definition of smooth ([**55**, IV. Définition (6.8.1)]), X and Y are regular. By lemma 3.77, X and Y are excellent. Now we apply proposition 3.78, which shows that $\tilde{f}^*$ is an isomorphism. By corollary 3.65, and since d was natural, we get that $f^* = d(X)^{-1} \circ \tilde{f}^* \circ d(Y) : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is an isomorphism.
The next proof independent of the characteristic is outlined in [**127**, 1.2.1] and based on [**111**, Proposition 10.5.]. By definition of birational, we have $k(Y) \cong k(X)$. By proposition 3.75 we have $\mathrm{Br}(Y) \cong \mathrm{Br}_{un}(k(Y)/k) \cong \mathrm{Br}_{un}(k(X)/k) \cong \mathrm{Br}(X)$. Since the inclusions involved in defining Br_{un} are functorial (cf. 3.74), the composition of those isomorphisms must equal f^*. $\square$

REMARK 3.81. In Grothendieck's [**49**, II, 2.] the arguments are similar to those in [**127**, 1.2.1] used in the proof of 3.80. Grothendieck worked with general schemes instead of projective k-varieties and could not use the unramified Brauer group, which would work in arbitrary dimension and characteristic, but required projectivity.

REMARK 3.82. Next to [**49**], [**127**] and [**111**], there are several other sources about the birational invariance of the Brauer group. Some use alternative approaches, e.g., [**89**, Theorem 42.5.] or [**72**, 8.3.].

Next we are going to state some results on the behavior of the Brauer group under special types of morphisms, which we need in the next chapter.

PROPOSITION 3.83. *Let k be a field, let $f : X \to Y$ be a generically finite dominant morphism of degree $d \in \mathbb{N}$ between smooth projective k-varieties, and assume that the field extension $k(X)/k(Y)$ is Galois. Then for $f^* = \mathrm{Br}(f) : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ we have $d \ker(f^*) \cong 0$.*
In particular if $n \in \mathbb{N}$ is coprime to d, and $\blacksquare[n]$ signifies the n torsion part of an abelian group, then the morphism $f^*[n] : \mathrm{Br}(Y)[n] \to \mathrm{Br}(X)[n]$ is injective.

[2]Just take $V := \mathbb{A}^2_{\mathbb{C}} \setminus \{(x, y); xy = 0\} \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$. Over $\mathbb{C}[x, 1/x, y, 1/y]$ the Azumaya algebra coming from the quaternion $\mathbb{C}(x, y)$-csas $(x, y)_2$ yields a nontrivial element in $H^2_{ét}(V, \mathbb{G}_m)$, while $H^2_{ét}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{G}_m) \cong 0$ (see [**22**, Example 12]).

PROOF. The proof proceeds along the lines of the proof of proposition 3.80. By dominance $k(X)/k(Y)$ is a field extension, and by generically finiteness we have that the field extension $k(X)/k(Y)$ is finite of degree d. By functoriality we have the commutative diagram

$$\begin{tikzcd} \mathrm{Br}(Y) \arrow[r, "f^*"] \arrow[d, hook] & \mathrm{Br}(X) \arrow[d, hook] \\ \mathrm{Br}(k(Y)) \arrow[r, "\mathrm{res}"] & \mathrm{Br}(k(X)), \end{tikzcd}$$

where res is the usual restriction map. Since the extension of function fields is Galois, we can directly apply relative Galois cohomology, which is a special type of group cohomology, and by [**117**, II.3. Proposition 9.] we get the restriction-corestriction sequence (or restriction-transfer sequence) $\mathrm{Br}(k(Y)) \xrightarrow{\mathrm{res}} \mathrm{Br}(k(X)) \xrightarrow{\mathrm{tr}} \mathrm{Br}(k(Y))$, which is equal to multiplication by d. The kernel of $\mathrm{tr} \circ \mathrm{res}$ is exactly $\mathrm{Br}(k(Y))[d]$, and $\ker(\mathrm{res}) \subset \mathrm{Br}(k(Y))[d]$ is a subgroup. The commutativity of the above diagram and the injectivity of the two vertical homomorphisms prove the claim. □

REMARK 3.84. In the separable but non-Galois case one could pass to the Galois cover and give an analogous argument. For the inseparable case cf. [**47**, Remarks 6.1.10 and 6.9.2] and [**117**, IV.2. Proposition 31.].

PROPOSITION 3.85. *Let k be a field, and let $f : X \to Y$ be a finite flat morphism of degree $d \in \mathbb{N}$ between smooth k-varieties which is Galois (cf. definition 3.41) with Galois group G. Then there is a G-action on $\mathrm{Br}(X)$ and we get $d^2 \,\mathrm{Br}(X)^G \subset f^* \,\mathrm{Br}(Y)$.*
In particular if $n \in \mathbb{N}$ is coprime to $d = |G|$, and $\blacksquare[n]$ signifies the n torsion part of an abelian group, then the morphism $f^*[n]' : \mathrm{Br}(Y)[n] \to \mathrm{Br}(X)[n]^G$ is bijective.

PROOF. See [**70**, Theorem 1.3]. □

PROPOSITION 3.86. *Let k be a field, and let $f : X \hookrightarrow Y$ be a dense open inclusion of regular quasi-projective k-varieties. Then $f^* = \mathrm{Br}(f) : \mathrm{Br}(Y) \hookrightarrow \mathrm{Br}(X)$ is an inclusion.*

PROOF. By corollary 3.65 we can substitute $H^2_{ét}$-groups for the Brauer groups. By the exact sequence in low degree terms for the Leray spectral sequence 3.59 with $\mathcal{F} \cong \mathbb{G}_m$ and $\pi = f$ it is enough to show $H^0_{ét}(X, R^1 f_* \mathbb{G}_m) \cong 0$. We need to show that the étale sheaf $R^1 f_* \mathbb{G}_m$ is 0, and by Hilbert's theorem 90 it suffices 3.23 to show vanishing for the Zariski sheaf $R^1 f_* \mathbb{G}_m$. By the sheaf properties, this is a local question, and we just need to show vanishing on the stalks. So we may assume that we have a regular noetherian local ring $R = \Gamma \mathcal{O}_{Y,y}$ and an open immersion $f' : U \hookrightarrow \mathrm{Spec}(R)$.
By [**61**, III. Proposition 8.5.] $R^1 f'_* \mathbb{G}_m$ is the associated sheaf of $H^1_{zar}(U, \mathbb{G}_m) \cong$

$\mathrm{Pic}(U)$. Regular implies locally factorial, and thus by [**61**, II. Corollary 6.16.] $\mathrm{Pic}(U)$ is isomorphic to $\mathrm{Cl}(U)$, the class group of Weil divisors. Since f' is an open immersion, the generic points are isomorphic, i.e., $k(U) \cong k(\mathrm{Spec}(R))$, and $\mathrm{Div}(U) \subset \mathrm{Div}(\mathrm{Spec}(R))$. Therefore linear equivalence on $\mathrm{Div}(U)$ is the restriction of linear equivalence on $\mathrm{Div}(\mathrm{Spec}(R))$. So $\mathrm{Pic}(U) \cong \mathrm{Cl}(U) \subset \mathrm{Cl}(\mathrm{Spec}(R))$. Since every regular noetherian local ring is a unique factorization domain we see by [**61**, II. Proposition 6.2.] that $\mathrm{Cl}(\mathrm{Spec}(R)) \cong 0$ and this concludes our argument.
Alternatively, since $k(X) \cong k(Y)$ one may use functoriality (proposition 3.39) to get $\mathrm{Br}(Y) \xrightarrow{\mathrm{Br}(f)} \mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(k(Y))$ is induced by $\mathrm{Spec}(k(Y)) \hookrightarrow Y$ and then proposition 3.63 to deduce $\alpha \circ \mathrm{Br}(f)$ is injective. Then $\mathrm{Br}(f)$ is also injective. $\square$

REMARK 3.87. Assume $\mathrm{char}(k) = 0$. For X, Y as above a closer examination using the theorem of Auslander and Brumer 3.68(4) leads to a refined version. For $Z \in \{X, Y\}$ denote the set of codimension 1 points by $\mathcal{R}_{1,Z}$ and the associated residue fields by κ_z. We get functorial exact sequences

$$0 \to H^2_{\acute{e}t}(Z, \mathbb{G}_m) \to H^2_{\acute{e}t}(\mathrm{Spec}(k(Z)), \mathbb{G}_m) \xrightarrow{\sum_{z \in \mathcal{R}_{1,Z}} \partial_{\Gamma \mathcal{O}_{Z,z}}} \bigoplus_{z \in \mathcal{R}_{1,Z}} \mathbf{X}(\mathrm{Gal}(\kappa_z^{sep}/\kappa_z)),$$

where the sum is well defined, since for a given $c \in H^2_{\acute{e}t}(\mathrm{Spec}(k(Z)), \mathbb{G}_m)$, we have $\partial_{\Gamma \mathcal{O}_{Z,z}}(c) = 0$ for almost all $z \in \mathcal{R}_{1,Z}$. This is a slight variation of the theorem of Auslander and Brumer exchanging factorial local ring spectra by regular varieties. Let $\mathcal{R}_{1,Y \setminus X} := \mathcal{R}_{1,Y} \setminus \mathcal{R}_{1,X}$. Comparing the sequences for $Z \in \{X, Y\}$ we arrive at

$$0 \to H^2_{\acute{e}t}(Y, \mathbb{G}_m) \to H^2_{\acute{e}t}(X, \mathbb{G}_m) \xrightarrow{\sum_{y \in \mathcal{R}_{1,Y \setminus X}} \partial_{\Gamma \mathcal{O}_{Y,y}}} \bigoplus_{y \in \mathcal{R}_{1,Y \setminus X}} \mathbf{X}(\mathrm{Gal}(\kappa_y^{sep}/\kappa_y)).$$

Cf. [**82**], which also features an example for computing images under residue maps, or [**22**, 4.].
This sequence is particularly interesting for computations when $\mathcal{R}_{1,Y \setminus X}$ is finite.

PROPOSITION 3.88. *Let $f : Y \to X$ be a finite flat and separated morphism of schemes such that $f_! = f_*$ on the étale site, $\forall i \in \mathbb{N} : R^i f_* = 0$ and $f^* \mathbb{G}_{m,X} \cong \mathbb{G}_{m,Y}$. Let $g : X' \to X$ be any morphism, then the following diagram exists and commutes:*

$$\begin{tikzcd} H^2_{\acute{e}t}(Y, \mathbb{G}_m) \arrow[r] \arrow[d, "\mathrm{cores}_{Y/X}"'] & H^2_{\acute{e}t}(Y \times_X X', \mathbb{G}_m) \arrow[d, "\mathrm{cores}_{Y \times_X X'/X'}"] \\ H^2_{\acute{e}t}(X, \mathbb{G}_m) \arrow[r, "\mathrm{Br}(g)"'] & H^2_{\acute{e}t}(X', \mathbb{G}_m) \end{tikzcd}$$

PROOF. Use [**4**, Exposeè XVII. Théorèm 6.2.3. (Var 2)] and then apply cohomology. The conditions on f are exactly such that we get the above diagram, taking into account that the trace map induces corestriction. $\square$

REMARK 3.89. We give an alternative argument for the functoriality property of 3.88. By the first half of the proof of [**70**, Theorem 1.4] we get for a finite flat morphism $f : Y \to X$ of noetherian schemes whose Brauer group and cohomological Brauer group are canonically isomorphic a corestriction map $\mathrm{cores}_{Y/X} : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ (this argument – valid in this more general context compared to that in loc. cit. – uses the existence of a norm map of structure sheaves, cf. [**55**, II (6.5.1)]). If we fit f into a fiber diagram of noetherian schemes as on the left the diagram we get a commutative square as on the right by compatibility of the norm map with base change for such f, of the Leray spectral sequence with pull backs (uses [**94**, II. Corollary 3.6.], i.e., $R^i f_* = 0$ for f finite and $i > 0$) and of the Leray spectral sequence with change of the coefficient sheaf which all carries over to cohomology. These facts are also used in the following section.

$$\begin{tikzcd} Y \arrow[d, "f"'] & Y' \arrow[l, "g'"'] \arrow[d, "f'"] \\ X & X' \arrow[l, "g"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Br}(Y) \arrow[r, "\mathrm{Br}(g')"] \arrow[d, "\mathrm{cores}_{Y/X}"'] & \mathrm{Br}(Y') \arrow[d, "\mathrm{cores}_{Y'/X'}"] \\ \mathrm{Br}(X) \arrow[r, "\mathrm{Br}(g)"'] & \mathrm{Br}(X') \end{tikzcd}$$

COROLLARY 3.90. *Let X be a quasi-projective smooth k-variety for k a number field, l/k a finite Galois extension and $k_{\mathfrak{p}}$ one of the local fields associated to k. Then the following two diagrams exist and commute:*

$$\begin{tikzcd} X_l \arrow[d] & \mathrm{Spec}(l \otimes_k k_{\mathfrak{p}}) \arrow[l] \arrow[d] \\ X & \mathrm{Spec}(k_{\mathfrak{p}}) \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Br}(X_l) \arrow[r] \arrow[d, "\mathrm{cores}_{X_l/X}"'] & \mathrm{Br}(l \otimes_k k_{\mathfrak{p}}) \arrow[d, "\mathrm{cores}_{l\otimes_k k_{\mathfrak{p}}/k_{\mathfrak{p}}}"] \\ \mathrm{Br}(X) \arrow[r] & \mathrm{Br}(k_{\mathfrak{p}}) \end{tikzcd} .$$

PROOF. The first diagram is simply the pullback diagram. We apply proposition 3.88 and check the conditions on $f : X_l \to X$. Since f is induced by a finite Galois field extension, it is finite flat separated and étale. By [**94**, VI. Theorem 3.2.] this implies $f_! = f_*$ and $\forall i \in \mathbb{N} : R^i f_* = 0$. By [**94**, II. Remarks 3.1.(d)] we also have $f^*\mathbb{G}_{m,X} \cong \mathbb{G}_{m,X_l}$, since f is étale and we work in the étale site. Now apply corollary 3.65 and we get the second commutative square. □

Now we turn to purity results, which we take from [**49**, III, 6.].

PROPOSITION 3.91. *Let X be a regular scheme and Y a closed subscheme of codimension $d \geq 2$. Let p be a prime and $\blacksquare[p]$ denote the p-power torsion part of an abelian group. Assume that one of the following conditions holds:*

(1) $\dim(X) \leq 2$, *or*
(2) X *is a smooth k-scheme and* $\mathrm{char}(k) \neq p$, *or*
(3) X *is excellent and of characteristic* 0.

Then the natural map $H^2_{ét}(X, \mathbb{G}_m)[p] \to H^2_{ét}(X \setminus Y, \mathbb{G}_m)[p]$ is an isomorphism.

PROOF. See [**49**, III, Corollaire (6.2)]. □

REMARK 3.92. This result is referred to as purity theorem, because it states that ramification can only occur in pure codimension 1. Cf. the well-known purity theorem for finite morphisms to a regular variety, which states that the ramification locus of such morphisms is of pure codimension 1. See [**53**, X. Théorèm 3.4].

DEFINITION 3.93. We call a birational morphism $f : X \to Y$ of integral schemes respecting codimension 1, if and only if it induces a bijection on the codimension 1 points of X and Y.

COROLLARY 3.94. *Let k be a field of characteristic 0, and let $f : X \to Y$ be a birational morphism of smooth quasi-projective k-varieties that respects codimension 1. Then $f^* = \mathrm{Br}(f) : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is an isomorphism.*

PROOF. Combine propositions 3.78, 3.91 and 3.65. □

COROLLARY 3.95. *Let k be a field, and let $f : X \to Y$ be a birational morphism of smooth quasi-projective k-varieties that respects codimension 1, and assume that either X or Y is projective. Then $f^* = \mathrm{Br}(f) : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is an isomorphism.*

PROOF. The arguments in the proof of proposition 3.80 rely only on facts about codimension 0 and 1 points, and projective is only invoked to get "enough" codimension 1 points. Thus $\mathrm{Br}(Y) \cong \mathrm{Br}_{un}(k(Y)/k) \cong \mathrm{Br}(X)$ yields by functoriality that f^* is an isomorphism. □

COROLLARY 3.96. *Let k be a field and X and Y smooth quasi-projective k-varieties that can be linked by a finite sequence of birational morphisms each either as in proposition 3.80, or as in corollary 3.95, to the same smooth projective k-variety Z. Then $\mathrm{Br}(X)$, $\mathrm{Br}(Y)$, $\mathrm{Br}_{un}(k(X)/k)$ and $\mathrm{Br}_{un}(k(Y)/k)$ are canonically isomorphic.*

PROOF. Immediate from proposition 3.80 and corollary 3.95. □

REMARK 3.97. In the literature results like the last are often stated under the hypothesis on the base field to have characteristic 0, because it is derived from results of Grothendieck in [**49**], which avoids projectivity conditions. Saltman in [**111**] showed how to use the unramfied Brauer group under a condition on projectivity to get results also valid in characteristic $p > 0$. This seems to be often overlooked, even when the primary concern is about projective varieties.

We also state another variant of the purity theorem, since it is of interest for the arithmetic of surfaces.

PROPOSITION 3.98. *If X is a regular scheme of $dim(X) \leq 3$ and $Y \subset X$ a closed subscheme of codimension $d \geq 2$, then $H^2_{\acute{e}t}(X, \mathbb{G}_m) \to H^2_{\acute{e}t}(X \setminus Y, \mathbb{G}_m)$ is an isomorphism.*

PROOF. See [**45**, Ch. I, Theorem 2'], which is slightly more general. □

REMARK 3.99. Let k be a number field and S a finite set of finite places of k. Let R be the ring of S-integers. A smooth arithmetic scheme of relative dimension 2 is a smooth scheme of relative dimension 2 over $\mathrm{Spec}(R)$ such that the fiber over the generic point of $\mathrm{Spec}(R)$ is an algebraic surface (variety of dimension 2), i.e., geometrically integral separated and of finite type over the number field k. As such it is 3-dimensional, and so the last purity theorem may be applicable.
For applications concerning only properties of Brauer groups or its elements relative to the fibers of the 1-dimensional scheme $\mathrm{Spec}(R)$ the purity results for schemes of dimension ≤ 2 suffice. E.g., in studying the ramification behavior of single Brauer group elements represented by a certain Azumaya algebra, as done in [**79**, 9.].

3.2.6. Galois Cohomology for Number Fields and Class Field Theory

Let k be a number field. We present in this section the Hasse reciprocity law and vanishing of $H^3_{gal}(k, k^*) \cong 0$. The results are valid in modified form for function fields of curves over finite fields (see [**20**, VII]). We use standard terminology from algebraic number theory, like the decomposition group of a prime for a Galois extension, without further notice. One may consult [**100**] or [**42**]. We use the notation of subsection 2.2.1.

These are results related to local and global class field theory, and we illuminate their background. We outline a proof of the reciprocity laws using duality theorems for cohomology. Good introductions to that subject can be found in [**100**], which keeps use of cohomology to a minimum, or [**20**], which first introduces Galois and Tate cohomology, and uses this language to formulate the results. Local class field theory is also treated with formulas in [**39**]. The idea is quite simple: as Galois theory relates Galois field extensions to groups, one wants to get a more explicit description of these groups in order to understand fields; class field theory does that job for abelian Galois groups in relating them to norm subgroups of the base field.

PROPOSITION 3.100. *Let G be a finite group, and let M be a $\mathbb{Z}$-free G-module and $M' := \mathrm{Hom}_{\mathbb{Z}-mod}(M, \mathbb{Z})$ with induced G-action. The cup product defines a non-degenerate pairing*

$$\hat{H}^i_{grp}(G, M') \times \hat{H}^{-i}_{grp}(G, M) \xrightarrow{\cup} \hat{H}^0_{grp}(G, \mathbb{Z}) \cong \frac{1}{\#G}\mathbb{Z}/\mathbb{Z},$$

and it induces an isomorphism of abelian groups $\hat{H}^i_{grp}(G, M') \cong \mathbf{X}(\hat{H}^{-i}_{grp}(G, M))$, where $\mathbf{X}(\cdot)$ denotes the dual defined by $\mathbf{X}(N) := \mathrm{Hom}_{cont}(N, \mathbb{Q}/\mathbb{Z})$, which is closely related to the Pontrayagin dual (see [**101**, pp. 7/325]*) and satisfies* ($N^{**} = N \Leftrightarrow$ *N is finite).*

PROOF. See [**101**, (7.2.1) Theorem]. □

REMARK 3.101. In 3.68 M could be an arbitrary topological group, while here we consider topological G-modules (with discrete topology).

PROPOSITION 3.102. *Let L/l be a finite Galois extension of local fields, and let M be a finitely generated $\mathbb{Z}$-free* $\mathrm{Gal}(L/l)$*-module and* $M' := \mathrm{Hom}_{\mathbb{Z}-mod}(M, L^*)$ *with induced* $\mathrm{Gal}(L/l)$*-action. Then the cup product defines a non-degenerate pairing*

$$\hat{H}^i_{grp}(\mathrm{Gal}(L/l), M') \times \hat{H}^{2-i}_{grp}(\mathrm{Gal}(L/l), M) \xrightarrow{\cup} H^2_{grp}(\mathrm{Gal}(L/l), L^*) \cong \frac{1}{\#\,\mathrm{Gal}(L/l)}\mathbb{Z}/\mathbb{Z},$$

and it induces an isomorphism of finite abelian groups $\hat{H}^i_{grp}(\mathrm{Gal}(L/l), M') \cong \mathbf{X}(\hat{H}^{2-i}_{grp}(\mathrm{Gal}(L/l), M))$.

PROOF. See [**101**, (7.2.1) Theorem]. □

COROLLARY 3.103. *Let L/l be a finite Galois extension of local fields. Then* $H^3_{grp}(\mathrm{Gal}(L/l), L^*) \cong 0$ *and* $H^3_{gal}(l, l^{alg\,*}) \cong 0$.

PROOF. Setting $M = \mathbb{Z}$ and $i = 3$ in the previous proposition and $G = \mathrm{Gal}(L/l), M = \mathbb{Z}$ and $i = -1$ in the second last proposition, we see that

$$H^3_{grp}(\mathrm{Gal}(L/l), L^*) \cong \hat{H}^3_{grp}(\mathrm{Gal}(L/l), \mathrm{Hom}(\mathbb{Z}, L^*)) \cong \mathbf{X}(\hat{H}^{-1}_{grp}(\mathrm{Gal}(L/l), \mathbb{Z})) \cong$$
$$\hat{H}^1_{grp}(\mathrm{Gal}(L/l), \mathrm{Hom}_{\mathbb{Z}-mod}(\mathbb{Z}, \mathbb{Z})) \cong H^1_{grp}(\mathrm{Gal}(L/l), \mathbb{Z}).$$

By definition of cohomology the last group is isomorphic to the group of continuous homomorphisms of the finite group $\mathrm{Gal}(L/l)$ into the free abelian group $\mathbb{Z}$. Due to elementary consideration on the order of group elements, there cannot be any nontrivial such homomorphisms, thus $H^3_{grp}(\mathrm{Gal}(L/l), L^*) \cong 0$.
The second statement follows then, since cohomology commutes with direct limits according to lemma 3.44. □

PROPOSITION 3.104. *Let L/l be a finite Galois extension of local fields. Then there is a canonical isomorphism of abelian groups*

$$r_{L/l} : \mathrm{Gal}(L/l)^{ab} \xrightarrow{\cong} l^*/N_{L/l}L^*$$

called the reciprocity isomorphism, where $G^{ab} := G/[G,G]$ denotes the abelianization of a group G and $N_{L/l}$ is the norm map for field extensions.

PROOF. The theorem is clear for archimedean local fields, and we assume l to be non-archimedean.
Setting $M = \mathbb{Z}$ and $i = 0$ in proposition 3.102, we get $\hat{H}^0_{grp}(\mathrm{Gal}(L/l), L^*) \cong \hat{H}^0_{grp}(\mathrm{Gal}(L/l), \mathrm{Hom}_{\mathbb{Z}-mod}(\mathbb{Z}, L^*)) \cong \mathbf{X}(\hat{H}^2_{grp}(\mathrm{Gal}(L/l), \mathbb{Z}))$.
We have $\mathbf{X}(\hat{H}^2_{grp}(\mathrm{Gal}(L/l), \mathbb{Z})) \cong \mathbf{X}(H^1_{grp}(\mathrm{Gal}(L/l), \mathbb{Q}/\mathbb{Z})) \cong \mathbf{X}(\mathbf{X}((\mathrm{Gal}(L/l)^{ab}))) \cong \mathrm{Gal}(L/l)^{ab}$. The first isomorphy follows from the long exact sequence in cohomology derived from $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and taking into account that $\mathbb{Q}$ is a uniquely divisible group as in the proof of proposition 3.63. $\mathbb{Q}/\mathbb{Z}$ is also divisible, but not uniquely. The second isomorphy is an easy exercise about the definition of cohomology groups (see, e.g., [**101**, p. 50]), and the third follows, because $\mathrm{Gal}(L/l)$ is finite by assumption.

We have $\hat{H}^0_{grp}(\mathrm{Gal}(L/l), L^*) \cong l^*/N_{L/l}L^*$ by definition of the 0-th Tate cohomology group (see [**20**, IV.6. p. 102]). $\square$

Definition 3.105. Using the terminology of the last proposition we define a map

$$(\cdot, L/l) : l^* \to \mathrm{Gal}(L/l)^{ab}, a \mapsto r_{L/l}(a \cdot N_{L/l}L^*)$$

and call it the local norm residue map (local norm residue symbol) for the Galois extension L/l.

This bijection between abelian quotient groups and certain subgroups of the multiplicative group determined by the norm map for a fixed Galois extension can be extended to the algebraic closure.

Proposition 3.106 (main theorem of local class field theory). *Let l be a local field. Then the map from the set of abelian extensions of l to (in the topology induced by the topology of the non-archimedian local field) open subgroups of the multiplicative group l^* defined by*

$$L \mapsto \mathcal{N}_L := N_{L/l}L^*$$

is a bijection. Moreover we have for L_1, L_2 abelian extensions of l:
$L_1 \subseteq L_2 \Leftrightarrow \mathcal{N}_{L_1} \supseteq \mathcal{N}_{L_2},\quad \mathcal{N}_{L_1L_2} = \mathcal{N}_{L_1} \cap \mathcal{N}_{L_2},\quad \mathcal{N}_{L_1 \cap L_2} = \mathcal{N}_{L_1}\mathcal{N}_{L_2}.$

Proof. The archimedian case is clear, for the non-archimedian case see [**100**, V. (1.4) Theorem]. $\square$

Next for K/k a finite Galois number field extension we look at the ideles and the defining sequence for the idele class group $1 \to K^* \to \mathbb{I}_K \to \mathrm{Cl}_{\mathbb{I}}(K) \to 1$. Denote primes of $\mathfrak{o}_K$ by $\mathfrak{P}$, and the prime of $\mathfrak{o}_k$ which $\mathfrak{P}$ extends by $\mathfrak{p}$.

Proposition 3.107. *In the above situation $\hat{H}^1_{grp}(\mathrm{Gal}(K/k), \mathrm{Cl}_{\mathbb{I}}(K)) \cong 0$, $H^1_{gal}(k, \varinjlim_{K/k} \mathrm{Cl}_{\mathbb{I}}(K)) \cong 0$ and $H^2_{gal}(k, \varinjlim_{K/k} \mathrm{Cl}_{\mathbb{I}}(K)) \cong \mathbb{Q}/\mathbb{Z}$.*

Proof. See [**101**, (8.1.12) Proposition and (8.1.13) Corollary] and [**20**, VII.11.2.(bis)]. $\square$

Proposition 3.108. *In the above situation $\hat{H}^3_{grp}(\mathrm{Gal}(K/k), K^*)$ is a finite cyclic group, and there is a finite Galois extension K'/K such that the inflation map (see* [**101**, I.5. p.45] *or 3.60)* inf : $\hat{H}^3_{grp}(\mathrm{Gal}(K/k), K^*) \to \hat{H}^3_{grp}(\mathrm{Gal}(K'/k), K'^*)$ *is the* 0*-map. It holds $H^3_{gal}(k, k^{alg\ *}) \cong 0$.*

Proof. We follow [**20**, VII.11.4. Case $r = 3$.]. For $i \in \mathbb{Z}$ we have: $\hat{H}^i_{grp}(\mathrm{Gal}(K/k), \mathbb{I}_K) \cong \bigoplus_{\mathfrak{p} \in \Omega_k} \hat{H}^i_{grp}(\mathrm{Gal}(K_{\mathfrak{P}}/k_{\mathfrak{p}}), K^*_{\mathfrak{P}})$ (see [**101**, (8.1.2) Proposition]). Consequently by corollary 3.103 $\hat{H}^3_{grp}(\mathrm{Gal}(K/k), \mathbb{I}_K) \cong 0$. In the long exact sequence for the idele class group sequence we get

$$\ldots \to \hat{H}^2_{grp}(\mathrm{Gal}(K/k), \mathbb{I}_K) \xrightarrow{\phi} \hat{H}^2_{grp}(\mathrm{Gal}(K/k), \mathrm{Cl}_{\mathbb{I}}(K)) \to$$
$$\hat{H}^3_{grp}(\mathrm{Gal}(K/k), K^*) \to \hat{H}^3_{grp}(\mathrm{Gal}(K/k), \mathbb{I}_K) \cong 0 \to \ldots$$

Thus we have $\hat{H}^3_{grp}(\mathrm{Gal}(K/k), K^*) \cong \mathrm{coker}(\phi)$.
Let $D := \hat{H}^0_{grp}(\mathrm{Gal}(K/k), \mathbb{Z}), C_{\mathfrak{p}} := \hat{H}^0_{grp}(\mathrm{Gal}(K_{\mathfrak{P}}/k_{\mathfrak{p}}), \mathbb{Z})), C := \prod_{\mathfrak{p} \in \Omega_k} C_{\mathfrak{P}}$, and $f : D \to C$ be the canonical morphism. Define $n := \#\,\mathrm{Gal}(K/k), n_{0,\mathfrak{p}} := \#\,\mathrm{Gal}(K_{\mathfrak{P}}/k_{\mathfrak{p}})$ and $n_0 := \mathrm{lcm}_{\mathfrak{p} \in \Omega_k}\{n_{0,\mathfrak{p}}\}$.
According to [**20**, VII.11.4] $\mathrm{coker}(\phi)$ is isomorphic to the dual of $\ker(f)$. This result is based on lengthy technical calculation, Tate's theorem (see [**20**, IV.10. Theorem 12]), and again duality arguments as above. By definition of 0-th Tate cohomology, D is isomorphic to $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ and $C_{\mathfrak{P}}$ to $\frac{1}{\#\,\mathrm{Gal}(K_{\mathfrak{P}}/k_{\mathfrak{p}})}\mathbb{Z}/\mathbb{Z}$, and since f maps the standard generators to the standard generators as careful inspection of the restriction map shows, we find $\hat{H}^3_{grp}(\mathrm{Gal}(K/k), K^*) \cong \frac{n_0}{n}\mathbb{Z}/\mathbb{Z}$.
Inspection of the inflation map yields that it is enough to find a $K'/K/k$ such that the decomposition group of some prime $\mathfrak{P}'$ over $\mathfrak{p}$ is divisible by n. Since Eisenstein polynomials may force an extension totally ramified at the respective prime, the decomposition group of totally ramified primes is the whole Galois group, and Eisenstein polynomials of arbitrary degree exist, this is possible. Of course there are many other extensions $K''/K/k$ satisfying the conditions.
Since $H^3_{gal}(k, k^{alg\,*})$ is the direct limit of the $\hat{H}^3_{grp}(\mathrm{Gal}(K/k), K^*)$ via the inflation maps, and since for each element in $\hat{H}^3_{grp}(\mathrm{Gal}(K/k), K^*)$ there is an inflation map sending it to 0, this implies that this direct limit must be 0. $\square$

Now recall the exact sequence of proposition 3.68 for local Henselian rings R with fraction field K and residue field k:

$$0 \to \mathrm{Br}(R) \to \mathrm{Br}(K) \xrightarrow{\partial_R} \mathbf{X}(\mathrm{Gal}(k^{sep}/k)) \to 0.$$

A non-archimedian local field l has an associated complete and hence Henselian ring $\mathfrak{o}_l$ (see, e.g., [**94**, I. Proposition 4.5.] and remark 2.58). By proposition 3.67 we know therefore that $\mathrm{Br}(\mathfrak{o}_l) \cong \mathrm{Br}(\kappa_l)$, where κ_l is the residue field, which is a finite field. By Wedderburn's little theorem (see, e.g., [**47**, p. 143]) there are no nontrivial (finite dimensional) division algebras over finite fields, and hence all csas are isomorphic to matrix algebras, so that $\mathrm{Br}(\kappa_l) \cong 0$. Galois groups of finite fields are well-known (see, e.g., [**100**, IV.2. Beispiel 5]): $\mathrm{Gal}(\kappa_l^{alg}/\kappa_l) \cong \hat{\mathbb{Z}}$. Thus by simple calculation (see, e.g., [**100**, IV.2. Beispiel 8] for the dual computation, which is easily reversed) $\mathbf{X}(\mathrm{Gal}(\kappa_l^{alg}/\kappa_l)) \cong \mathbb{Q}/\mathbb{Z}$. Combining all this we get a short exact sequence

$$0 \to 0 \to \mathrm{Br}(l) \xrightarrow{\mathrm{inv}_l} \mathbb{Q}/\mathbb{Z} \to 0.$$

Definition 3.109. The isomorphism of abelian groups

$$\mathrm{inv}_l : \mathrm{Br}(l) \to \mathbb{Q}/\mathbb{Z}$$

just defined is called the local invariant map for (the Brauer group of) the local field l.
For an l-csa $\mathcal{A}$ we call $\mathrm{inv}_l([\mathcal{A}])$ the invariant (Hasse invariant) of $\mathcal{A}$.
By the results on the Brauer groups of $\mathbb{R}$ and $\mathbb{C}$ (example 3.19) we also get unique embeddings of the associated Brauer groups into $\mathbb{Q}/\mathbb{Z}$ denoted by $\mathrm{inv}_{\mathbb{R}}$ and $\mathrm{inv}_{\mathbb{C}}$.

REMARK 3.110. Usually the invariant map is defined more directly instead of using the cohomological machinery. See [**116**, XIII. Proposition 6.] or [**20**, V.1. Appendix and VII.11.].
The invariant map has an explicit inverse, which is obvious for $l \in \{\mathbb{R}, \mathbb{C}\}$ from example 3.19 and otherwise is defined as follows (see [**75**, 13.10 Satz])

$$\psi : \mathbb{Q}/\mathbb{Z} \to \mathrm{Br}(l), \frac{m}{n} + \mathbb{Z} \mapsto [(l_n, F_n, \pi^m)].$$

The terminology for cyclic csas for unramified extension of local fields is analogous to example 3.17.

REMARK 3.111. The arguments above did not require characteristic 0, i.e., this result on the invariant map holds for the function field case as well.

PROPOSITION 3.112 (Hasse reciprocity law). *For a number field k we have the following commutative diagram with exact rows:*

$$\begin{tikzcd}
 & 0 \arrow[r] & \bigoplus_{\mathfrak{p}\in\Omega_k} \mathrm{Br}(k_\mathfrak{p}) \arrow[r, "\cong"] \arrow[d, "\mathrm{id}"] & \bigoplus_{\mathfrak{p}\in\Omega_k^0} \mathbb{Q}/\mathbb{Z} \oplus \bigoplus_{\mathfrak{p}\in\Omega_k^\mathbb{R}} \frac{1}{2}\mathbb{Z}/\mathbb{Z} \oplus \bigoplus_{\mathfrak{p}\in\Omega_k^\mathbb{C}} \mathbb{Z}/\mathbb{Z} \arrow[r] \arrow[d, "s:=\sum_{\mathfrak{p}\in\Omega_k} s_\mathfrak{p}"] & 0 \\
0 \arrow[r] & \mathrm{Br}(k) \arrow[r] & \bigoplus_{\mathfrak{p}\in\Omega_k} \mathrm{Br}(k_\mathfrak{p}) \arrow[r, "\mathrm{inv}=\sum_{\mathfrak{p}\in\Omega_k} \mathrm{inv}_\mathfrak{p}"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

PROOF. Combining propositions 3.108 and 3.107 we get from the long exact sequence derived from the defining sequence of the idele class group after taking direct limits over all Galois extensions the short exact sequence

$$H^1_{gal}(k, \varinjlim_{K/k} \mathrm{Cl}_\mathbb{I}(K)) \cong 0 \to H^2_{gal}(k, k^{alg\ *}) \to H^2_{gal}(k, \mathbb{I}_{k^{alg}}) \to$$
$$H^2_{gal}(k, \varinjlim_{K/k} \mathrm{Cl}_\mathbb{I}(K)) \to 0 \cong H^3_{gal}(k, k^{alg\ *}).$$

By corollary 3.65 for $X = \mathrm{Spec}(k)$ and lemma 3.44, we have $H^2_{gal}(k, k^{alg\ *}) \cong \mathrm{Br}(k)$. Using the same argument again and additionally the relation between cohomology for local fields and global fields with idele coefficients as in [**20**, VII.11.4] or [**101**, (8.1.7) Proposition], we get $H^2_{gal}(k, \mathbb{I}_{k^{alg}}) \cong \oplus_{\mathfrak{p}\in\Omega_k} H^2_{gal}(k_\mathfrak{p}, k_\mathfrak{p}^{alg\ *}) \cong \oplus_{\mathfrak{p}\in\Omega_k} \mathrm{Br}(k_\mathfrak{p})$. With proposition 3.107 we have $H^2_{gal}(k, \varinjlim_{K/k} \mathrm{Cl}_\mathbb{I}(K)) \cong \mathbb{Q}/\mathbb{Z}$.

Thus using definition 3.109, the inclusions $k \hookrightarrow k_\mathfrak{p}$ and functoriality of cohomology, we get a commutative diagram as stated in the proposition. It remains only to show that the morphisms are as claimed, particularly that inv indeed is $\sum_{\mathfrak{p}\in\Omega_k} \mathrm{inv}_\mathfrak{p}$. To this end it is enough to show that the first vertical morphism is truly the identity and that all the $s_\mathfrak{p}$ are the standard inclusions into $\mathbb{Q}/\mathbb{Z}$. This means going through all the many isomorphisms, and cohomology sequences we used for the construction and we omit these technical considerations. See [**101**, (8.1.17) Theorem]. $\square$

For completeness we state the results on the reciprocity map of global class field theory in the case of number fields. There is also an obvious global norm residue map, but contrary to the local version, we do not need it and leave it undefined.

PROPOSITION 3.113. *Let K/k be a finite Galois extension of number fields. Then there is a canonical isomorphism of abelian groups*

$$\mathrm{Gal}(K/k)^{ab} \xrightarrow{\cong} \mathrm{Cl}_{\mathbb{I}}(k)/N_{K/k}\,\mathrm{Cl}_{\mathbb{I}}(K)$$

called the reciprocity isomorphism, where $G^{ab} := G/[G,G]$ denotes the abelianization of a group, and $N_{K/k}$ is the norm map for field extensions.

PROOF. See [**100**, VI. (5.5) Theorem]. □

PROPOSITION 3.114 (main theorem of global class field theory for number fields). *Let k be a number field. Then the map from the set of finite abelian extensions of k to closed subgroups of finite index of the idele class group $\mathrm{Cl}_{\mathbb{I}}(k)$ defined by*

$$K \mapsto \mathcal{N}_K := N_{K/k}\,\mathrm{Cl}_{\mathbb{I}}(K)$$

is bijective. Moreover we have for K_1, K_2 finite abelian extensions of k:
$K_1 \subseteq K_2 \Leftrightarrow \mathcal{N}_{K_1} \supseteq \mathcal{N}_{K_2}$, $\mathcal{N}_{K_1K_2} = \mathcal{N}_{K_1} \cap \mathcal{N}_{K_2}$, $\mathcal{N}_{K_1 \cap K_2} = \mathcal{N}_{K_1}\mathcal{N}_{K_2}$.

PROOF. See [**100**, VI. (6.1) Theorem]. □

REMARK 3.115. We have an isomorphism $\mathrm{Br}(\mathbb{A}_k) \xrightarrow{\sim} \oplus_{\mathfrak{p}\in\Omega_k} \mathrm{Br}(k_{\mathfrak{p}})$ compatible with 3.112 (see [**135**, Remark 3]). It is straightforward to see from the reference that this isomorphism is natural in k and is compatible with inclusions and projections $k_{\mathfrak{p}} \hookrightarrow \mathbb{A}_k \twoheadrightarrow k_{\mathfrak{p}}$ from 2.64 after applying the Brauer functor.

3.2.7. A Filtration of the Brauer Group of a Variety

In this subsection let k be a number field although the first few results are true for arbitrary fields if one replaces ^{alg} by ^{sep}, and let X be a smooth projective k-variety. Smoothness and quasi-projectivity are needed to exploit the isomorphism between the Azumaya Brauer group and the group $H^2_{\acute{e}t}(X, \mathbb{G}_m)$, and projectivity is needed for $\Gamma(X, \mathcal{O}_X^*) \cong k^*$. We exploit these properties immediately:

PROPOSITION 3.116. *Let K be a finite Galois extension of k, and denote $G :=$ $\mathrm{Gal}(K/k)$. Then we have a (functorial) exact sequence*

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_K)^G \to H^2_{grp}(G, K^*) \to$$
$$\ker(\mathrm{Br}(X) \to \mathrm{Br}(X_K)) \to H^1_{grp}(G, \mathrm{Pic}(X_K)) \to H^3_{grp}(G, K^*).$$

There is another (functorial) exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X^{alg})^{\mathrm{Gal}(k^{alg}/k)} \to \mathrm{Br}(k) \to$$
$$\ker(\mathrm{Br}(X) \to \mathrm{Br}(X^{alg})) \to H^1_{grp}(\mathrm{Gal}(k^{alg}/k), \mathrm{Pic}(X_{k^{alg}})) \to 0.$$

PROOF. Essentially this is proposition 3.58 with $\mathcal{F} = \mathbb{G}_m$. Let π denote the fiber product morphism $X_K \to X$.
By definition of étale cohomology (cf. [**124**, II. (4.3)]), we have $H^0_{ét}(X_K, \mathbb{G}_m) \cong \Gamma(X_K, \mathcal{O}^*_{X_K}) \cong K^*$, where the last isomorphism holds because X_K is a projective variety. The first term is then $H^1_{grp}(G, H^0_{ét}(X_K, \pi^*\mathbb{G}_m)) \cong H^1_{grp}(G, H^0_{ét}(X_K, \mathbb{G}_m)) \cong H^1_{grp}(G, K^*)$, which is the trivial group due to Hilbert's theorem 90 3.23 the 3rd variant.
The second term is $H^1_{ét}(X, \mathbb{G}_m)$, which by Hilbert's theorem 90 3.23 the 1st variant is $\mathrm{Pic}(X)$.
Similarly the third term is $H^0_{grp}(G, H^1_{ét}(X_K, \pi^*\mathbb{G}_m)) \cong H^0_{grp}(G, \mathrm{Pic}(X_K)) \cong \mathrm{Pic}(X_K)^G$.
The next term is $H^2_{grp}(G, H^0_{ét}(X_K, \pi^*\mathbb{G}_m)) \cong H^2_{grp}(G, K^*)$. This group is isomorphic to $\ker(\mathrm{Br}(k) \to \mathrm{Br}(K))$, where the map is the functorial one by lemma 3.44 and corollary 3.62.
Since X is smooth, X is regular, and since number fields are perfect, X is geometrically regular, hence X_K is regular. Thus in the same way as for the forth term the fifth term is $\ker(H^2_{ét}(X, \mathbb{G}_m) \to H^0_{grp}(G, H^2_{ét}(X_K, \pi^*\mathbb{G}_m))) \cong \ker(\mathrm{Br}(X) \to H^0_{grp}(G, \mathrm{Br}(X_K))) \cong \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_K)^G)$ by proposition 3.63. Since composition with an embedding like $\mathrm{Br}(X_K)^G \hookrightarrow \mathrm{Br}(X_K)$ does not change the kernel, we have $\ker(\mathrm{Br}(X) \to \mathrm{Br}(X_K)^G) \cong \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_K))$.
For the sixth term we have $H^1_{grp}(G, H^1_{ét}(X_K, \pi^*\mathbb{G}_m)) \cong H^1_{grp}(G, H^1_{ét}(X_K, \mathbb{G}_m)) \cong H^1_{grp}(G, \mathrm{Pic}(X_K))$ due to Hilbert's theorem 90 3.23 the 1st variant.
Finally using some of the arguments for the first term, the seventh term computes to $H^3_{grp}(G, H^0_{ét}(X_K, \pi^*\mathbb{G}_m)) \cong H^3_{grp}(G, K^*)$.
Taking the direct limit over all Galois extensions of k and using proposition 3.108 which gives $H^3_{gal}(k, k^{alg\ *}) \cong 0$ yields immediately the second sequence. $\square$

REMARK 3.117. The ideas of the last proposition can also be used to get similar results for other fields than number fields. The vanishing of certain cohomology groups used to get the precise statement above may not hold, and one has to carry the appropriate terms in more general situations. See 3.129 below for such an application.

REMARK 3.118. The application we have in mind is about k-rational points on X. The existence of k-rational points on X implies everywhere local solubility (see remark 2.79). According to the discussion of the last chapter this is an effectively testable hypothesis, and we may assume that X satisfies everywhere local solubility to draw further conclusions. Since X is projective proposition 2.69 tells us that everywhere locally solvable, i.e., $\prod_{\mathfrak{p} \in \Omega_X} X(k_\mathfrak{p}) \neq \emptyset$, is equivalent to X having adelic points, i.e., $X(\mathbb{A}_k) \neq \emptyset$.

Proposition 3.119. *If additionally to the conditions and notions of the last proposition we have $X(\mathbb{A}_k) \neq \emptyset$, then the second sequence decomposes into two parts:*

$$0 \to \operatorname{Pic}(X) \xrightarrow{\phi\ \cong} \operatorname{Pic}(X^{alg})^{\operatorname{Gal}(k^{alg}/k)} \xrightarrow{0}$$

$$\operatorname{Br}(k) \xrightarrow{\psi} \ker(\operatorname{Br}(X) \to \operatorname{Br}(X^{alg})) \to H^1_{grp}(\operatorname{Gal}(k^{alg}/k), \operatorname{Pic}(X^{alg})) \to 0.$$

This means that $\ker(\operatorname{Br}(X) \to \operatorname{Br}(X^{alg}))/\operatorname{Br}(k) \cong H^1_{grp}(\operatorname{Gal}(k^{alg}/k), \operatorname{Pic}(X^{alg})) = H^1_{gal}(k, \operatorname{Pic}(X^{alg}))$.

Proof. It suffices to show that ψ is an inclusion. Since ψ is induced by the functorial map $\operatorname{Br}(k) \xrightarrow{\psi'} \operatorname{Br}(X)$ coming from the structure map $X \xrightarrow{b} \operatorname{Spec}(k)$, it suffices to show that ψ' is an inclusion.
By remark 3.118 we have $\prod_{\mathfrak{p}\in\Omega_k} k_{\mathfrak{p}}$-points on X, i.e., there is morphism $\operatorname{Spec}(\prod_{\mathfrak{p}\in\Omega_k} k_{\mathfrak{p}}) \to X$. Composing this morphism with the structure morphism we get

$$\operatorname{Spec}(\prod_{\mathfrak{p}\in\Omega_k} k_{\mathfrak{p}}) \to X \xrightarrow{b} \operatorname{Spec}(k),$$

which is induced by the standard inclusions of k into its associated local fields (at least up to a Galois automorphism of k for each $\mathfrak{p}$, which can be reversed). Applying the Br-functor to this composition, we get the inclusion morphism of the Hasse reciprocity law 3.112. This inclusion decomposes by functoriality as

$$\operatorname{Br}(k) \xrightarrow{\psi'} \operatorname{Br}(X) \to \bigoplus_{\mathfrak{p}\in\Omega_k} \operatorname{Br}(k_{\mathfrak{p}}).$$

Since the composition is injective, ψ' is injective, which gives the desired inclusion. □

The last propositions suggest a filtration of the Brauer group of a smooth projective k-variety X, which has particularly nice properties when X satisfies everywhere locally solubility. This is not only meaningful for number fields, but generally for global fields.

Definition 3.120. Let k be a number field, and X a smooth projective k-variety. From the exact sequence above, we get canonical morphisms $\operatorname{Br}(k) \xrightarrow{\iota_0} \ker(\operatorname{Br}(X) \to \operatorname{Br}(X^{alg})) \xrightarrow{\iota_1} \operatorname{Br}(X)$, of which ι_1 is always an inclusion. We define

(1) $\operatorname{im}(\iota_1 \circ \iota_0) =: \operatorname{Br}_0(X) \subset \operatorname{Br}(X)$ to be the constant (trivial) part of the Brauer group,
(2) $\operatorname{im}(\iota_1) =: \operatorname{Br}_1(X) \subset \operatorname{Br}(X)$ to be the algebraic (arithmetic) part of the Brauer group.

An $[\mathcal{A}] \in \operatorname{Br}(X)$ is called constant, if and only if $[\mathcal{A}] \in \operatorname{Br}_0(X)$, it is called algebraic, if and only if $[\mathcal{A}] \in \operatorname{Br}_1(X) \setminus \operatorname{Br}_0(X)$, and it is called transcendental (geometric), if and only if $[\mathcal{A}] \in \operatorname{Br}(X) \setminus \operatorname{Br}_1(X)$.

This defines a filtration which is non-degenerate in general, but may degenerate in special cases:

$$\{\iota_0([k])\} \subset \mathrm{Br}_0(X) \subset \mathrm{Br}_1(X) \subset \mathrm{Br}(X).$$

By abuse of notation we call the quotients $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ the algebraic part and $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ the transcendental part. In the case when ι_0 is an inclusion, e.g., when X is everywhere locally solvable, we identify $\mathrm{Br}_0(X) = \mathrm{Br}(k)$.

REMARK 3.121. Assume X satisfies everywhere local solubility. Since the Hasse reciprocity law gives an explicit description of $\mathrm{Br}(k)$, the constant part of the Brauer group is well understood. Unfortunately as described in section 3.3, this part cannot give rise to any interesting arithmetic phenomena.
The last statement in proposition 3.119 can now be restated as

$$\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \cong \mathrm{Br}_1(X)/\mathrm{Br}(k) \cong H^1_{gal}(k, \mathrm{Pic}(X^{alg})).$$

The algebraic part of the Brauer group of a variety X can be understood in terms of its Picard group and Galois cohomology. In the case of diagonal quartics this route has been taken by Bright in [**13**]. Since group cohomology and the Picard group are fairly well understood mathematical concepts this may seem a simple task, but the technical details to make this isomorphism effectively computable are nontrivial. A shorter discussion in a less concrete setting may also be found in [**73**, III.4.] and in [**79**].
The effective methods described in [**79**] require X to satisfy certain conditions, e.g., a given finite presentation of $\mathrm{Pic}(X^{alg})$ with explicitly given divisors representing generators. In general the problem should not be considered solved, and where it is solved in principle practical considerations such as time and memory restrictions on computers may make it hard to actually compute the algebraic part of the Brauer group.
To understand the transcendental part of the Brauer group we need some knowledge of $\mathrm{Br}(X^{alg})$. The following proposition is in this direction.

PROPOSITION 3.122. *Let $l = l^{alg} \subset \mathbb{C}$ be an algebraically closed field, and Y a proper regular l-scheme. Let Y^{an} be the associated compact analytic manifold (cf. remark 2.43). Then* $H^2_{ét}(Y, \mathbb{G}_m) = \mathrm{Br}'(Y) \cong (\mathbb{Q}/\mathbb{Z})^{\mathrm{rk}\, H^2_{sing}(Y^{an},\mathbb{Z}) - \mathrm{rk}\,\mathrm{NS}(Y)} \oplus H^3_{sing}(Y^{an}, \mathbb{Z})_{tors}$.

PROOF. See [**73**, II.8.5. Proposition]. □

REMARK 3.123. In the case of surfaces a similar, slightly weaker result which is characteristic free is given by [**94**, V. Corollary 3.28.]. Obviously it does not involve analytic manifolds.

3.2.8. Fibrations and the Brauer Group of a Surface

PROPOSITION 3.124. *Let $f : X \to C$ define a fibered surface over a number field k, and let $\tilde{f} : \tilde{X} \to \tilde{C}$ be the associated fibration after base change to k^{alg}. Let $\tilde{\eta} \hookrightarrow \tilde{C}$ be the generic point and $\tilde{X}_{\tilde{\eta}} := \tilde{X} \times_{\tilde{C}} \tilde{\eta} \hookrightarrow \tilde{X}$ the generic fiber of $\tilde{f}$. As in 2.109 we* *denote* $\mathrm{Pic}_{vert}(\tilde{X}) := \ker(\mathrm{Pic}(\tilde{X}) \twoheadrightarrow \mathrm{Pic}(\tilde{X}_{\tilde{\eta}}))$. *Use the analogous definitions for the version without* $\tilde{}$. *By 3.73 and 3.75 we have the inclusion* $\mathrm{Br}(X) \xrightarrow{\iota} \mathrm{Br}(k(X))$, *and we also have* $\mathrm{Br}(k(C)) \xrightarrow{\mathrm{Br}(f)} \mathrm{Br}(k(X))$. *Define* $\mathcal{B} := (\iota^{-1}(\mathrm{im}(\mathrm{Br}(f))))/\,\mathrm{Br}(k)$. *Then*

$$\mathcal{B} \cong H^1_{gal}(k, \mathrm{Pic}_{vert}(\tilde{X})) \to H^1_{gal}(k, \mathrm{Pic}(X)) \cong \mathrm{Br}_1(X)/\,\mathrm{Br}(k).$$

Proof. See [**13**, Proposition 4.21.], where this is proved for the case $\tilde{C} \cong \mathbb{P}^1_{k^{alg}}$, which generalizes immediately, since the only specific thing about $\mathbb{P}^1_{k^{alg}}$ used is Tsen's theorem for its function field (see 3.21), which holds for any curve $\tilde{C}$. □

Remark 3.125. By this proposition we may do computations in the Brauer group of the function field of a curve $\mathrm{Br}(k(C))$, within which it is much easier to compute (see [**13**, 4.4.1]) compared to the Brauer group of a surface, and still get algebraic Brauer group elements for a surface. The map to the algebraic part needs neither be surjective, nor injective. We might not get Azumaya algebra representatives for all of the algebraic part of the Brauer group by this method.

We consider the use of fibrations to compute representatives for potentially transcendental classes. Since transcendental classes are those not annihilated when passing to the algebraic closure of the base field, we need results on algebraically closed base fields.

Proposition 3.126. *Let* $g : Y \to D$ *define a fibered surface over an algebraically closed field* $l = l^{alg}$. *Then*

$$\mathrm{Br}(Y) \cong H^1_{\acute{e}t}(D, \underline{\mathrm{Pic}}_{Y/D}),$$

where $\underline{\mathrm{Pic}}_{Y/D}$ *denotes the relative Picard sheaf for the étale topology as defined in* [**38**].

Proof. We use corollary 3.59, according to which we have with $\mathcal{F} = \mathbb{G}_m$ the exact sequence

$$H^2_{\acute{e}t}(D, R^0 g_* \mathbb{G}_m) \to \ker(H^2_{\acute{e}t}(Y, \mathbb{G}_m) \to H^0_{\acute{e}t}(D, R^2 g_* \mathbb{G}_m)) \to \\ H^1_{\acute{e}t}(D, R^1 g_* \mathbb{G}_m) \to H^3_{\acute{e}t}(D, R^0 g_* \mathbb{G}_m).$$

By remark 2.105 we may apply[3] [**38**, Exercise 9.3.11.] to get $R^0 g_* \mathbb{G}_a = g_* \mathbb{G}_a \cong \mathbb{G}_a$. Since $\mathbb{G}_m$ defines an étale subsheaf of sets of $\mathbb{G}_a$, we have $R^0 g_* \mathbb{G}_m \cong \mathbb{G}_m$ as étale sheaves on D. By [**38**, Remark 9.2.11.] $R^1 g_* \mathbb{G}_m \cong \underline{\mathrm{Pic}}_{Y/S}$.
By corollary 3.65 it holds that $H^2_{\acute{e}t}(Y, \mathbb{G}_m) \cong \mathrm{Br}(Y)$. Thus it suffices to prove the triviality of the following three groups: $H^0_{\acute{e}t}(D, R^2 g_* \mathbb{G}_m), H^2_{\acute{e}t}(D, \mathbb{G}_m), H^3_{\acute{e}t}(D, \mathbb{G}_m)$. The first one is trivial, since $R^2 g_* \mathbb{G}_m \cong 0$ by [**49**, III, Corollaire 3.2.] taking into account remark 2.105.
By proposition 3.75 and corollary 3.65, we have $H^2_{\acute{e}t}(D, \mathbb{G}_m) \cong \mathrm{Br}(D) \cong \mathrm{Br}_{un}(k(D)/l) \subset \mathrm{Br}(k(D))$, and the last of these groups is trivial due to Tsen's theorem 3.21.

[3] The terminology there is explained in [**38**, p. 251 and Exercise 9.2.5.].

$H^3_{ét}(D, \mathbb{G}_m)$ is trivial simply for dimension reasons and since l is algebraically closed, see, e.g., [**94**, VI. Theorem 1.1.].
Cf. [**94**, p. 218] for a variant of this result. $\square$

PROPOSITION 3.127. *In the situation of the last proposition let $n \in \mathbb{N}$ with* $\mathrm{char}\, l$ *and n coprime. If we denote the n-torsion part of (sheaves of) abelian groups by $\blacksquare[n]$, then we have the following diagram:*

$$H^1_{ét}(D, R^1 g_* \mu_n) \cong H^1_{ét}(D, \underline{\mathrm{Pic}}_{Y/D}[n]) \twoheadrightarrow H^1_{ét}(D, \underline{\mathrm{Pic}}^0_{Y/D})[n] \twoheadrightarrow H^1_{ét}(D, \underline{\mathrm{Pic}}_{Y/D})[n] \cong \mathrm{Br}(Y)[n]$$

PROOF. We follow [**80**, pp. 2/3]. The first isomorphism is more or less by definition (see the proof of [**94**, V. Theorem 3.5.] for an explicit related statement). By remark 2.113 g admits multisections. Assume that d is the greatest common divisor of the degrees of all multisections of g. By [**27**, 4.3] or [**18**, 8.2. Theorem 2 and 9.3. Theorem 1] we have a short exact sequence of étale sheaves

$$0 \to \underline{\mathrm{Pic}}^0_{Y/D} \to \underline{\mathrm{Pic}}_{Y/D} \to \mathbb{Z} \to 0 \tag{3.2.1}$$

and part of the derived long exact sequence reads

$$\begin{gathered} H^0_{ét}(D, \underline{\mathrm{Pic}}_{Y/D}) \xrightarrow{\phi} H^0_{ét}(D, \mathbb{Z}) \to \\ H^1_{ét}(D, \underline{\mathrm{Pic}}^0_{Y/D}) \to H^1_{ét}(D, \underline{\mathrm{Pic}}_{Y/D}) \to H^1_{ét}(D, \mathbb{Z}) \end{gathered}$$

We have $H^1_{ét}(D, \mathbb{Z}) \cong 0$ (e.g., see [**94**, p. 117]), $H^0_{ét}(D, \mathbb{Z}) \cong \mathbb{Z}$ (global sections of a constant sheaf are a power of that constant group to the number of connected components, and D is connected), and $\mathrm{im}(\phi) \cong d\mathbb{Z}$ (any global section of $\underline{\mathrm{Pic}}_{Y/D}$ defines a multisection and vice versa). Thus we get a short exact sequence

$$0 \to d\mathbb{Z}/\mathbb{Z} \to H^1_{ét}(D, \underline{\mathrm{Pic}}^0_{Y/D}) \to H^1_{ét}(D, \underline{\mathrm{Pic}}_{Y/D}) \to 0. \tag{3.2.2}$$

Cf. [**49**, III, (4.36)].
Now consider the multiplication by n morphism of sheaves $\underline{\mathrm{Pic}}^0_{Y/D} \xrightarrow{n\cdot} \underline{\mathrm{Pic}}^0_{Y/D}$. According to [**94**, proof of III. Corollary 4.19.(b)], since n is coprime to the characteristic of k, this is fiberwise surjective. Then using [**55**, IV. Corollair (21.9.12)] and [**38**, Proposition 9.5.19.], we see that this extends to the (étale) stalks and hence this morphism is surjective. On the other hand the kernel of $n\cdot$ is by definition $\underline{\mathrm{Pic}}^0_{Y/D}[n]$, and since by equation (3.2.1) the torsion of $\underline{\mathrm{Pic}}_{Y/D}$ is already included in $\underline{\mathrm{Pic}}^0_{Y/D}$, we have $\underline{\mathrm{Pic}}^0_{Y/D}[n] = \underline{\mathrm{Pic}}_{Y/D}[n]$ giving rise to the following short exact sequence of sheaves:

$$0 \to \underline{\mathrm{Pic}}_{Y/D}[n] \to \underline{\mathrm{Pic}}^0_{Y/D} \xrightarrow{n\cdot} \underline{\mathrm{Pic}}^0_{Y/D} \to 0.$$

We have as part of the long exact sequence in cohomology for this sequence

$$H^1_{ét}(D, \underline{\mathrm{Pic}}_{Y/D}[n]) \to H^1_{ét}(D, \underline{\mathrm{Pic}}^0_{Y/D}) \xrightarrow{H^1_{ét}(D, n\cdot)} H^1_{ét}(D, \underline{\mathrm{Pic}}^0_{Y/D}),$$

which yields

$$H^1_{ét}(D, \underline{\mathrm{Pic}}_{Y/D}[n]) \twoheadrightarrow H^1_{ét}(D, \underline{\mathrm{Pic}}^0_{Y/D})[n].$$

Composing this morphism of abelian groups with that of (3.2.2) and the isomorphism of the last proposition, and applying $\blacksquare[n]$ proves the proposition. $\square$

Remark 3.128. Let k be a number field and X a k-surface, and denote its base change to the algebraic closure k^{alg} by Y. If one wants to find transcendental elements of $\mathrm{Br}(X)$, then one is looking for elements in $\mathrm{Br}(X) \setminus (\ker(\mathrm{Br}(X) \to \mathrm{Br}(Y)))$. Since the morphism $\mathrm{Br}(X) \to \mathrm{Br}(Y)$ is induced by an algebraic field extension, we therefore look for nontrivial elements in $\mathrm{Br}(Y)$ that have an Azumaya algebra representative that descents to the original number field k.
The above proposition gives a method to identify (the n-torsion part of) $\mathrm{Br}(Y)$. This gives us candidates to test for descent, provided we specify a fibration $g : Y \to D$. This approach is treated in [**80**, 4.].

Proposition 3.129. *Let l be a perfect field and let Z be an l-curve. Then there is a functorial inclusion*

$$\mathrm{Br}(Z)/\mathrm{Br}_0(Z) \hookrightarrow H^1_{gal}(l, \mathrm{Pic}(Z^{alg})).$$

If $H^3_{gal}(l, l^{alg\,*})) \cong 0$, then this inclusion is an isomorphism.

Proof. Since l is perfect, we have $l^{sep} = l^{alg}$. The first part of proposition 3.116 is also valid for general fields not just for number fields, and thus taking the limit over all Galois extensions of l we get an isomorphism $\mathrm{Br}_1(Z)/\mathrm{Br}_0(Z) \cong \ker(H^1_{gal}(l, \mathrm{Pic}(Z^{alg})) \to H^3_{gal}(l, l^{alg\,*}))$. Since Z is a curve we have furthermore that $\mathrm{Br}(Z^{alg}) \cong 0$ by Tsen's theorem 3.21 and thus $\mathrm{Br}_1(Z) \cong \ker(\mathrm{Br}(Z) \to \mathrm{Br}(Z^{alg})) \cong \mathrm{Br}(Z)$. ☐

Remark 3.130. Let $f : X \to C$ define a fibered surface over a number field k, take the base change to the algebraic closure k^{alg}, and let $\mathrm{Spec}(l) = \eta \hookrightarrow \mathcal{C}$ be the generic point of the base changed base curve. Let $Z := \tilde{X} \times_{\tilde{C}} \eta \hookrightarrow \tilde{X}$ be the generic fiber of f, i.e., we have the pull back diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & \tilde{X} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\eta \arrow[r] & \tilde{C} \arrow[r] \arrow[d] & C \arrow[d] \\
& \mathrm{Spec}(k^{alg}) \arrow[r] & \mathrm{Spec}(k).
\end{tikzcd}$$

Now let $\iota : Z \to X$ be the induced morphism. ι induces a morphism of Brauer groups $\mathrm{Br}(X) \xrightarrow{\mathrm{Br}(\iota)} \mathrm{Br}(Z)$. l is the function field of a curve over an algebraically closed field namely k^{alg}, and thus by [**117**, IV.3. Corollary 1], and using Tsen's theorem from example 3.21, we get $H^3_{gal}(l, l^{alg\,*})) \cong 0$. Thus the inclusion analogous to the one in proposition 3.129 is an isomorphism. Again by Tsen's theorem $\mathrm{Br}(l) \cong 0$ thus $\mathrm{Br}_0(Z) \cong 0$. Combining this information, we get:

$$\mathrm{Br}(X) \xrightarrow{\mathrm{Br}(\iota)} \mathrm{Br}(Z) \xrightarrow{\cong} H^1_{gal}(l, \mathrm{Pic}(Z^{alg})).$$

By definition $\mathrm{Br}(\iota)$ annihilates algebraic elements of $\mathrm{Br}(X)$ since ι factors over $\tilde{X}$. In order to find nontrivial transcendental elements in $\mathrm{Br}(X)$, we may compute $H^1_{gal}(l, \mathrm{Pic}(Z^{alg}))$, and try to lift elements in this group to $\mathrm{Br}(X)$ via $\mathrm{Br}(\iota)$.
One can try to do this via an intermediate step using $\mathrm{Br}(\tilde{X}) \hookrightarrow \mathrm{Br}(Z)$ (injectivity is due to the fact, that the locus on which an Azumaya algebra of the function field can ramify gets smaller; cf. subsection 3.2.4). Assume we have an Azumaya algebra $\mathcal{A}$ representing a class in $\mathrm{Br}(Z)$. $\mathcal{A}$ represents also a class in the common function field of Z and $\tilde{X}$, and the question, whether it represents an Azumaya algebra in $\mathrm{Br}(\tilde{X})$, can be addressed using residue maps and the unramified Brauer group as in definition 3.73 applied to $\mathcal{A}$ (or a suitable constructed Brauer equivalent Azumaya algebra). It remains to decide, whether $\mathcal{A}$ or some Brauer equivalent Azumaya algebra descents to k, similarly as discussed in remark 3.128.

This method is applied in Ieronymou's paper [**69**], which explicitly computes transcendental 2-torsion in the Brauer group of a diagonal quartic. In this paper he relates $H^1_{gal}(l, \mathrm{Pic}(Z^{alg}))[2]$ to $H^1_{gal}(l, \mathrm{Jac}(Z)[2])$ using the type map for torsors (see [**120**]), and this last group is amenable to computations.

3.3. The Brauer-Manin Obstruction

For this section we assume k to be a number field. For a recent example of the use of Brauer-Manin obstruction for (generalizations of) global fields see a paper of Poonen and Voloch [**106**]. X is in most of this section a smooth projective k-variety.

In this situation we know by proposition 2.69, that

$$X(k) \xrightarrow{\iota_X} X(\mathbb{A}_k) \cong X(\prod_{\mathfrak{p}\in\Omega_k} k_{\mathfrak{p}}) \cong \prod_{\mathfrak{p}\in\Omega_k} X(k_{\mathfrak{p}}),$$

where ι_X is injective. We identify $\mathrm{im}(\iota_X)$ with $X(k)$ and also the later three sets.

We are interested in the set $X(k)$, but in general it is hard and maybe impossible (depending on the still unanswered question about the decidability of Hilbert's 10th problem for number fields) to compute $X(k)$ in general.

As we have seen in subsection 2.2.3 we can determine if $X(\mathbb{A}_k) = \emptyset$ effectively. In particular: $X(\mathbb{A}_k) = \emptyset \Rightarrow X(k) = \emptyset$. We may therefore assume that X is everywhere locally solvable (els), i.e., $X(\mathbb{A}_k) \neq \emptyset$.

We rephrase the question: how big is the gap in $X(k) \subset X(\mathbb{A}_k)$? Recalling definition 2.71, we explicitly ask whether X or a certain class of varieties $\mathbf{X}$ containing X satisfies HP, WA or SA, of which the later two are the same in the projective case that we study according to proposition 2.73. In other words we ask, if $X(k) = X(\mathbb{A}_k)$ (HP), or if they are the same at least up to topology, i.e., $\overline{X(k)}^{\mathbb{A}} = X(\mathbb{A}_k)$ (WA=SA). Summarizing we want to get more knowledge about the chain of inclusions

$$X(k) \subset \overline{X(k)}^{\mathbb{A}} \subset X(\mathbb{A}_k).$$

DEFINITION 3.131. For k a number field and X a k-variety (not necessarily smooth or projective) we say that X respectively any class of varieties $\mathbf{X}$ containing X admits

(1) an obstruction to existence of k-rational points, when we can show that $\emptyset = X(k)$,
(2) an obstruction to HP, when we can show that $\emptyset = X(k) \subsetneq X(\prod_{\mathfrak{p}\in\Omega_k} k_{\mathfrak{p}})$,
(3) an obstruction to WA, when we can show that $\overline{X(k)}^{prod} \subsetneq X(\prod_{\mathfrak{p}\in\Omega_k} k_{\mathfrak{p}})$.
(4) an obstruction to SA, when we can show that $\overline{X(k)}^{\mathbb{A}} \subsetneq X(\mathbb{A}_k)$.

REMARK 3.132. Assume X is projective and has els. An obstruction to existence of k-rational points is equivalent to an obstruction to HP, and both imply an obstruction to WA, which is equivalent to an obstruction to SA. In this situation an obstruction to WA may occur despite the impossibility of an obstruction to HP.

Usually for smooth projective X satisfying els such obstructions are proved by finding an effectively computable set U with

$$X(k) \subset \overline{X(k)}^{\mathbb{A}} \subset U \subset X(\mathbb{A}_k).$$

If $U = \emptyset$, we have an obstruction to HP, and if $U \subsetneq X(\mathbb{A}_k)$, we have an obstruction to WA.

The Brauer-Manin obstruction is a method to produce such a set U called a Brauer-Manin set. Depending on whether the U produced by this method is empty or just properly included in $X(\mathbb{A}_k)$, we say that we have a Brauer-Manin obstruction to HP or WA, respectively.

In order to understand the Brauer-Manin obstruction we need to introduce the Brauer-Manin pairing defined by an evaluation map.

LEMMA 3.133. *For a smooth k-variety X and a place $\mathfrak{p} \in \Omega_k$ we define*

$$\mathrm{ev}_{\mathfrak{p}} : \mathrm{Br}(X) \times X(k_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}, ([\mathcal{A}], p : \mathrm{Spec}(k) \to X) \mapsto \mathrm{inv}_{k_{\mathfrak{p}}}(\mathrm{Br}(p)([\mathcal{A}]))$$

and

$$\mathrm{ev} : \mathrm{Br}(X) \times X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}, ([\mathcal{A}], p : \mathrm{Spec}(\mathbb{A}_k) \to X) \mapsto \sum_{\mathfrak{p}\in\Omega_k} \mathrm{ev}_{\mathfrak{p}}([\mathcal{A}], p \circ \pi_{\mathfrak{p}}),$$

where $\pi_{\mathfrak{p}}$ is the canonical morphism $\mathrm{Spec}(\mathbb{A}_k) \to \mathrm{Spec}(k_{\mathfrak{p}})$.
Then ev *is well-defined, in that all except a finite number of summands in the sum to define* ev *are* $0 \in \mathbb{Q}/\mathbb{Z}$.
Both maps induce group homomorphisms when the second component is fixed, and continuous morphisms when the first component is fixed where we take the discrete topology on $\mathbb{Q}/\mathbb{Z}$.
Both maps induce constant maps when the first component is fixed in $\mathrm{Br}_0(X)$, *and* ev *induces the* 0*-map in this situation.* ev *also induces the* 0*-map when the second component is a fixed element* $x \in X(k) \subset X(\mathbb{A}_k)$.

PROOF. We only sketch how to prove that ev is well-defined. More details can be found in [**73**, III. Proposition 2.3. and Lemmata 3.4./3.5.] or in [**79**, 9.].
In subsection 2.2.3 we discussed presentations of k-varieties. In proposition 2.84 we gave a result on how smooth projective k-varieties admit a projective arithmetic model $\mathfrak{X}$ over $\mathrm{Spec}(o_k)$, which is smooth and projective away from a finite set of places S' and remarked that an analogous statement holds in the non-projective case.
Let $p \in X(\mathbb{A}_k)$ be fixed. Locally at $\mathrm{im}(p) \in U$ the Brauer class $[\mathcal{A}]\,|_U$ can be represented uniquely by its restriction to the generic point of the open affine subvariety U by definition 3.73 and proposition 3.75, which by remark 3.18 can be represented as a crossed product $k(U)$-csa, i.e., by a Galois extension $E/k(U)$ with Galois group G, and a G-2-cocycle $\Phi : G^2 \to E^*$ (see [**103**, 14.1.]). To deal with ramification several but finitely many representations might be necessary, for simplicity we assume only one representation suffices.
This is a finite amount of data involving finitely many coefficients in $k(U)$ and ultimately finitely many coefficients in k by a finite type argument. Except at finitely many non-archimedian places S'', where the rational functions defining the cocycle or the field extension vanish entirely or are a pole entirely, this data induces a Galois field extension $E_{\mathfrak{p}}/k(U)_{\mathfrak{p}}$ and a valid 2-cocycle defining away from S' a csa on the function field of $\mathfrak{U}_{\mathfrak{o}_{\mathfrak{p}}}$. This set S'' can be determined effectively. Represent the rational functions as quotients of polynomials $(n_i/d_i)_{i\in I}$ in the local coordinates of U. It suffices to determine the set of primes at which numerators n_i and denominators d_i vanish completely, i.e., if $\{f_1, \dots, f_m\}$ are local equations for U as a subvariety of $\mathbb{A}^n$ when the ideal generated by the f_j and a single $1-n_i$ or $1-d_i$ generates 1, which can be checked by effective Hilbert's Nullstellensatz based on Gröbner basis computation. As discussed previously Buchberger's algorithm can detect primes of non-general behavior like ramification at additional codimension 1 subvarieties. These are only finitely many and form the set S''. In general this method will not be very efficient, but effective. By repeating the process with other representatives one might be able to restrict S'' even further.
For the archimedian places we only have to worry about the real ones, since the invariant at the complex ones is always vanishing.
When precomposing p with the various $\pi_{\mathfrak{p}}$, then in all but finitely many cases the morphism does factor over $\mathrm{Spec}(\mathfrak{o}_{\mathfrak{p}})$ by definition of adelic points. Collect the exceptional $\mathfrak{p}$ in the finite set $S''' \subset \Omega_k$. So whenever $\mathfrak{p} \in \Omega \setminus S$ we have a factorization $\mathrm{Spec}(k_{\mathfrak{p}}) \to \mathrm{Spec}(\mathfrak{o}_{\mathfrak{p}}) \to X$ for p.
Let $S := S' \cup S'' \cup \Omega_k^{\mathbb{R}} \cup S'''$ and for $\mathfrak{p} \notin S$ let $p_{\mathfrak{p}}$ be the associated component of p, which by choice of S is even a $\mathfrak{o}_{\mathfrak{p}}$-point. Using the commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(k_{\mathfrak{p}}) \arrow[rr, "p_{\mathfrak{p}}"] \arrow[d] & & X \arrow[d] \\
\mathrm{Spec}(\mathfrak{o}_{\mathfrak{p}}) \arrow[r, "p'_{\mathfrak{p}}"] & \mathfrak{X}_{\mathfrak{o}_{\mathfrak{p}}} \arrow[r] & \mathfrak{X}
\end{tikzcd}$$

we get away from S by functoriality of the involved constructions like unramified Brauer groups (here we use smoothness), a factorization of the induced homomorphism of Brauer groups. $\mathrm{Br}(X) \to \mathrm{Br}(\mathrm{Spec}(\mathfrak{o}_\mathfrak{p})) \to \mathrm{Br}(\mathrm{Spec}(k_\mathfrak{p}))$ sends $[\mathcal{A}]$ to the specialization of $[\mathcal{A}_\mathfrak{p}]$ via the lower part, and then to the specialization of $[\mathcal{A}]$ via $p_\mathfrak{p}$. According to proposition 3.67, $\mathrm{Br}(\mathrm{Spec}(\mathfrak{o}_\mathfrak{p})) \cong \mathrm{Br}(\mathrm{Spec}(\kappa_\mathfrak{p}))$ and since $\kappa_\mathfrak{p}$ is a finite field we get by 3.21 that $\mathrm{Br}(\mathrm{Spec}(\mathfrak{o}_\mathfrak{p})) \cong 0$. The sum $\sum_{\mathfrak{p}\in\Omega_k} \mathrm{ev}_\mathfrak{p}([\mathcal{A}], p\circ\pi_\mathfrak{p}) = \sum_{\mathfrak{p}\in S} \mathrm{ev}_\mathfrak{p}([\mathcal{A}], p\circ\pi_\mathfrak{p})$ is finite, and ev is well defined.
The group homomorphism property is immediate, since the $\mathrm{inv}_{k_\mathfrak{p}}$ are group homomorphisms (cf. definition 3.109 and the discussion before that).
As discussed before, Brauer classes are trivialized over $\mathrm{Spec}(\mathfrak{o}_\mathfrak{p})$ and since the translates of $\mathfrak{o}_\mathfrak{p}$ in $k_\mathfrak{p}$ are open sets (see, e.g., [**100**, II.5.]), and the topology on $X(k_\mathfrak{p})$ is induced by the topology on $k_\mathfrak{p}$, the map $\mathrm{ev}_\mathfrak{p}$ for fixed first component is locally constant hence continuous, since $\mathbb{Q}/\mathbb{Z}$ is furnished with the discrete topology. By componentwise arguments and the definition of the adelic topology, this gives the analogous result for ev.
Let $p, p' \in X(k_\mathfrak{p})$ for some $\mathfrak{p} \in \Omega_k$. There might be different embeddings $k \hookrightarrow k_\mathfrak{p}$ as abstract field due to field automorphisms of k, but the choice of $\mathfrak{p}$ fixes that. Hence p, p' induce the same morphism b to $\mathrm{Spec}(k)$ after composition with the structure morphism. The pullbacks of $[\mathcal{A}] \in \mathrm{Br}_0(X)$ via p and p' give the same Brauer group element, since both are the pullback of a Brauer element over $\mathrm{Spec}(k)$ via b. Summing all the invariants for an adelic point we chase a Brauer group element of $\mathrm{Br}(\mathrm{Spec}(k))$ through the short exact sequence of the Hasse reciprocity law 3.112, and therefore get $0 \in \mathbb{Q}/\mathbb{Z}$. Thus ev induces the 0-map, if the first component is fixed to an element of $\mathrm{Br}_0(k)$. By the same argument ev induces the 0-map, if the second component is a fixed adelic point induced by a k-point. $\square$

Definition 3.134. In the above situation the map $\mathrm{ev}_\mathfrak{p}$ is called local evaluation map at $\mathfrak{p}$, and ev global evaluation map, or Brauer-Manin pairing.
For a fixed $[\mathcal{A}] \in \mathrm{Br}(X)$ we get an induced map $\mathrm{ev}_{[\mathcal{A}]} : X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}, x \mapsto \mathrm{ev}([\mathbb{A}], x)$ called the evaluation map for $[\mathcal{A}]$. The evaluation map for $[\mathcal{A}]$ at a place $\mathfrak{p}$ is defined in the analog way and denoted $\mathrm{ev}_{\mathfrak{p},[\mathcal{A}]}$.

In the proof of the last lemma we show, that the summands in the definition of ev are non-zero only at a finite set of places $S_{([\mathcal{A}],x)}$ for a given pair $([\mathcal{A}], x)$ depending on x. The advantage is that one does not need any properness or projectivity conditions on X. We now give a uniform bound on the nontrivial places, i.e., an $S_{[\mathcal{A}]}$ depending only on the Brauer group element.

Lemma 3.135. *For a smooth projective k-variety X and a fixed $[\mathcal{A}] \in \mathrm{Br}(X)$, there is a finite set $S \subset \Omega_k$, such that $\mathrm{ev}_{\mathfrak{p},[\mathcal{A}]} = 0$ is the 0-map.*

Proof. In the projective case we may multiply the coordinates of any $x \in X(\mathbb{A}_k)$ until all components over finite primes are actually integral. Then $S''' = \emptyset$ in the proof of lemma 3.133 and S is independent of x. $\square$

REMARK 3.136. The above proof was ineffective insofar as we have chosen a given crossed product representation for $[\mathcal{A}]$. If we already have a crossed product representation or a representation as a cyclic algebra $\mathcal{A}$, then the set S is effectively determined by $S' \cup S'' \cup \Omega_k^{\mathbb{R}}$, i.e., places of bad reduction of X, places of bad reduction for the representing sheaf of Azumaya algebras and the real places.

REMARK 3.137. One can generalize to proper and dismiss smooth in the conditions, and get an analogous result (see [**73**, III. Proposition 2.3. and Lemmata 3.4./3.5.]). However the involved arguments do not seem to give S effectively.

PROPOSITION 3.138. *Let X be a k-variety and $\mathcal{B} \subset \mathrm{Br}(X)$. The set $X(\mathbb{A}_k)^{\mathcal{B}} := \{x \in X(\mathbb{A}_k) : \forall [\mathcal{A}] \in \mathcal{B} : \mathrm{ev}([\mathcal{A}], x) = 0\}$ is a subset of $X(\mathbb{A}_k)$ closed relative to the adelic topology and contains $X(k)$, i.e.*

$$X(k) \subset \overline{X(k)}^{\mathbb{A}} \subset X(\mathbb{A}_k)^{\mathcal{B}} \subset X(\mathbb{A}_k).$$

PROOF. We can write $X(\mathbb{A}_k)^{\mathcal{B}} = \bigcap_{[\mathcal{A}] \in \mathcal{B}} \mathrm{ev}_{[\mathcal{A}]}^{-1}(0)$. Since by lemma 3.133 all $\mathrm{ev}_{[\mathcal{A}]}$ are continuous and $0 \in \mathbb{Q}/\mathbb{Z}$ is closed in the discrete topology, $X(\mathbb{A}_k)^{\mathcal{B}}$ is an intersection of closed sets hence closed. By the same lemma all adelic points induced by k-rational points evaluate to 0. This proves the claim. □

DEFINITION 3.139. With the notation as in the last proposition we call $X(\mathbb{A}_k)^{\mathcal{B}}$ the Brauer set of $X(k)$ for $\mathcal{B} \subset \mathrm{Br}(X)$.
If $X(\mathbb{A}_k)^{\mathcal{B}}$ proves one of the obstructions in definition 3.131, we say that $\mathcal{B}$ gives rise to a Brauer-Manin obstruction for X or any class $\mathbf{X}$ containing X.

REMARK 3.140. Again by lemma 3.133 the elements of $\mathrm{Br}_0(X)$ are irrelevant for Brauer-Manin obstructions. Thus constant Brauer elements may be neglected.
Lemma 3.133 also shows, that the Brauer-Manin set of a $\mathcal{B} \subset \mathrm{Br}(X)$ and for the subgroup of $\mathrm{Br}(X)$ generated by it, are the same.

REMARK 3.141. One can use the Brauer-Manin obstruction also the other way: if we are unsure, whether a Brauer class $[\mathcal{A}]$ given by any concrete description of a representative, e.g., by cocycle data, is constant, we may compute $X(\mathbb{A}_k)^{\{[\mathcal{A}]\}}$, and if it is smaller than $X(\mathbb{A}_k)$, we know that $[\mathcal{A}]$ is non-constant.

REMARK 3.142. If we have an arithmetic model of the k-variety X we can think of it as varying with "geometric parameters" and an "arithmetic parameter". For the geometric part imagine a $\mathbb{C}$-manifold. The arithmetic parameter shows up, e.g., in that the Krull dimension of $\mathbb{Z}$ is 1. The Brauer-Manin obstruction can be viewed as a tool to combine geometrically varying objects (non-constant Brauer elements represented by sAas) with arithmetically varying ones (adelic points) via the Hasse reciprocity law 3.112 to give restrictions on the actual points.

REMARK 3.143. There are also other methods to construct obstructions, and they are usually more powerful, but harder to compute. There is the fundamental obstruction (see [**120**, p. 34]), the descent obstruction (see [**120**, 5.3]), which defines also a subset of the adelic points similar to Brauer set,

$$X(k) \subset \overline{X(k)}^{\mathbb{A}} \subset X(\mathbb{A}_k)^{desc} \subset X(\mathbb{A}_k),$$

and the étale Brauer-Manin obstruction (see [**105**]), which defines a Brauer like set

$$X(k) \subset \overline{X(k)}^{\mathbb{A}} \subset X(\mathbb{A}_k)^{\acute{e}t\ Br} \subset X(\mathbb{A}_k).$$

We have the following chain of inclusions for X a smooth projective k-variety:

$$X(k) \subset \overline{X(k)}^{\mathbb{A}} \subset X(\mathbb{A}_k)^{desc} = X(\mathbb{A}_k)^{\acute{e}t\ Br} \subset X(\mathbb{A}_k)^{\mathrm{Br}(X)} \subset X(\mathbb{A}_k).$$

The seminal paper [**118**] of Skorobogatov proves that the second last inclusion can be proper. The middle inclusion to the right can be found in Skorobogatov's paper [**119**], and the converse can be found in a recent preprint of Demarche [**34**]. An example to show that the second inclusion is proper, can be found in a recent preprint of Poonen [**105**]. For a discussion on the relation of descent obstruction and the fundamental obstruction see a preprint of Harari and Stix [**58**].
Recently Harpaz and Schlank outlined a homotopy obstruction in [**59**] with the potential to be more restrictive than any previous obstruction and Schlank in [**112**] explained Poonen's example by a ramified étale Brauer-Manin obstruction.

Definition 3.144. If for one of the obstruction sets U mentioned or defined above, we can prove for a class $\mathbf{X}$ of varieties that $\forall X \in \mathbf{X} : X(k) = U$, we say that the obstruction type defining U is the only one for $\mathbf{X}$.

Remark 3.145. Obviously by remark 3.143 neither Brauer-Manin obstruction, nor the more elaborate ones except possible the new obstructions by Harpaz and Schlank can be used as definitive decision procedure for the question of existence of rational points, i.e., the analog of Hilbert's 10th problem for number fields. But one may ask for classes of varieties for which this question can be decided using Brauer-Manin obstruction.
Coliot-Thélène, Sansuc and Sir Swinnerton-Dyer showed in a series of papers [**25**], that Brauer-Manin obstruction is the only one for Châtelet surfaces. And Coliot-Thélène, Skorobogatov and Sir Swinnerton-Dyer showed in [**27**] that under Schinzel's hypothesis, which is a still unproven conjecture the Brauer-Manin obstruction is the only one, provided there is a fibration of the given variety to $\mathbb{P}^1$ satisfying certain conditions. For a curve E of genus 1 with finite Tate-Shafarevitch group $Ш_1(E)$, which by the Birch and Swinnerton-Dyer conjecture should always be the case, Brauer-Manin obstruction is the only one (see [**120**, p. 114]). There are also similar conjectures for more general curves (see [**120**, p. 127]), and Coliot-Thélène conjectured that for rationally connected smooth projective k-varieties Brauer-Manin obstruction is the only one (see [**24**]).
There have been given several examples of Brauer-Manin obstruction since its invention. To our knowledge for surfaces over $\mathbb{Q}$ these were all mediated by algebraic elements whose order is 2- or 3-primary or by transcendental 2-torsion elements.

In [**78**, Example 7] a Brauer-Manin obstruction to HP is given which cannot be explained by a single Brauer element. Specifically for the surface considered there at the place 17 for each of the three generators of the algebraic part (the transcendental part is trivial) there are points with local invariant 0 but at each point exactly two of these generators have invariant $\frac{1}{2}$. From this we see that none of the 7 nontrivial Brauer elements explains failure of HP on its own. So one needs to consider sometimes more than one element (or equivalently cyclic subgroups) to achieve most general obstruction. Note that there is also obstruction to WA at other places in this example, but even combined information from all these places for a single Brauer element cannot explain failure of HP.
For a fixed Brauer element and a fixed place different global rational points may have different invariants, as [**64**, 7.3.] shows by example. I.e., at a single place a Brauer element might partition the local solutions in classes such that local solutions which are part of global solutions are properly partitioned. This potential partition might have some arithmetic meaning yet to be understood.
To avoid misconceptions we clearly state here, that the order of an obstructing Brauer group and the places at which it ramifies, i.e., has nontrivial invariant, can a priori occur in arbitrary combinations. In particular a p-torsion element can but does not have to have trivial invariant at the place p for a prime p.

3.4. Brauer-Manin Obstruction for K3 Surfaces

In his notes for an advanced introductory course on non-abelian descent published in [**29**] Harari gives a nice overview of the recent standing of the power of the Brauer-Manin obstruction in light of the classification of surfaces. We discussed some results in that direction at the end of chapter 2.

As discussed in subsection 2.3.2 K3 surfaces are the surfaces at the edge of surfaces with widely believed conjectures for their qualitative arithmetic behavior and the abyss of mostly unexplored types of surfaces. The experts in the field do not agree on what to expect. For example while Harari in [**29**] writes that he has "no clear idea whether the Brauer-Manin obstruction should be the only one" for K3 surfaces, Skorobogatov in a talk in Lousanne in 2009 expressed a vague feeling that it should be the only one, and Sir Swinnerton-Dyer in a talk in Zürich in 2010 just did the contrary. He also considered this problem a hard one, calling it a research problem for a graduate student, who is extremely gifted and one extremely dislikes at the same time.

But there is also considerable progress in recent years. Skorobogatov and Zarhin in [**121**, Theorem 1.2.] proved unconditionally, that the Brauer group modulo the constant part $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ of a K3 surface X defined over a finitely generated field over $\mathbb{Q}$ like the number field k, is finite. Their proof relies on results of Faltings on the Tate conjecture for divisors, which are inherently ineffective. We thus do not get bounds, that could be useful in computations.

On the other hand Kresch and Tschinkel in [**80**, Theorem 1.] gave a method to effectively compute for a given $n \in \mathbb{N}$ a set $\mathcal{B}$ which satisfies

$$(\mathrm{Br}(X)/Br_0(K))[n] \subset \mathcal{B} \subset \mathrm{Br}(X)/Br_0(K)$$

based on proposition 3.127, and then go on to give a method to effectively compute the associated Brauer set denoted $X(\mathbb{A}_k)^{\mathcal{B}}$ which satisfies

$$X(\mathbb{A}_k)^{\mathrm{Br}(X)} \subset X(\mathbb{A}_k)^{\mathcal{B}} \subset X(\mathbb{A}_k)^{\mathrm{Br}(X)[n]},$$

provided that the smooth projective k-surface X has torsion free geometric Picard group $\mathrm{Pic}(X^{alg})$ and both are given explicitly enough. Apart from the explicitly enough condition a K3 surface satisfies these conditions a priori according to proposition 2.44.

Kresch and Tschinkel in [**79**, Theorem 3.4.] get under the same conditions, that $\mathrm{Br}_1(X)/\,\mathrm{Br}_0(X)$ is effectively computable, as is $X(\mathbb{A}_k)^{\mathrm{Br}_1(X)}$.

Combining all these results one needs only to bound the order of transcendental elements up to algebraic elements, and one can compute the Brauer set for the whole Brauer group in the case of K3 surfaces. Such bounds seem to be currently out of reach in general.

The methods described by Kresch and Tschinkel depend on X to satisfy els. With the application on rational points in mind this is however not a real restriction as discussed in remark 3.118. On the other hand if one is keen on computing $\mathrm{Br}(X)[n]/Br_0(K)$ and $\mathrm{Br}_1(X)/\,\mathrm{Br}_0(X)$ in the general case, this should be a straight forward generalization using the more complicated exact sequence in proposition 3.116 instead of the simpler one of proposition 3.119.

We now discuss the progress in terms of examples and then come to diagonal quartics.

The first example for a Brauer-Manin obstruction on a K3 surfaces over $\mathbb{Q}$ facilitated by a transcendental Brauer element seems to be given in Wittenberg's paper [**131**]. The Brauer group element is given in the form of a quaternion algebra and is therefore 2-torsion. He relies on an elliptic fibration for this particular K3 surface.

Very recently a Hassett, Várilly-Alvarado and Varilly in [**64**] gave an example of transcendental Brauer-Manin obstruction to WA on a special K3 surface X over $\mathbb{Q}$, that is a double cover of the projective plane branched in a sextic curve. They show, that it has geometric Picard rank 1, and therefore by proposition 2.52 can not have an elliptic fibration, not even geometrically. The usual method of using elliptic fibrations could not be applied to their surface. Instead they used a link between 2-torsion elements of the Picard group of a K3 surface and cubic 4-folds in $\mathbb{P}^5$. This link is facilitated by projective space bundles giving rise to elements in $\check{H}^1_{ét}(X, \underline{\mathrm{PGl}}_n)$ for some n, and lemma 3.29 shows how this relates to the Brauer group of X. The used sheaves of Azumaya algebras are 2-torsion.

For diagonal quartic K3 surfaces over $\mathbb{Q}$ Bright in his thesis [**13**] and several follow up papers discussed the algebraic part of their Brauer group. He gave effective methods and implementations in `MAGMA` for their computation, and gave examples of computations of obstructions to the Hasse principle. Some of the results and their proofs discussed in chapter 2 concerning diagonal quartics can also be found there. Bright listed all possible groups that might occur as $\operatorname{Br}_1(X)/\operatorname{Br}_0(X)$, and thus came up with a bound

$$\#(\operatorname{Br}_1(X)/\operatorname{Br}_0(X))|2^5.$$

Ieronymou gave for diagonal quartics X over $\mathbb{Q}$ an explicit description of the 2-torsion in $\operatorname{Br}(X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k^{alg})) \cong (\mathbb{Q}/\mathbb{Z})^2$ in [**69**], and gave condition when they descent to nontrivial elements in $\operatorname{Br}(X)[2]$. His results were also based on an elliptic fibration. The case of transcendental 2-torsion is thus more or less completed.

Shortly after Ieronymou, Skorobogatov and Zarhin in [**70**] gave a bound on the transcendental Brauer group of diagonal quartics over $\mathbb{Q}$ namely

$$\forall[\mathcal{A}] \in \operatorname{Br}(X)/\operatorname{Br}_1(X) : \operatorname{ord}([\mathcal{A}])|2^{10} \cdot 3 \cdot 5.$$

Together with Bright's bound for the algebraic part this yields

$$\#(\operatorname{Br}(X)/\operatorname{Br}_0(X))|2^{25}3^2 5^2$$

and thus by the results of Kresch and Tschinkel discussed above the Brauer-Manin obstruction of diagonal quartics over $\mathbb{Q}$ is effectively computable. In [**70**] they also give a simple criterion that implies $\operatorname{Br}_1(X) = \operatorname{Br}(X)$, i.e., that there are no transcendental elements in the Brauer group. The proof of these results is based on the fact, that diagonal quartics at least geometrically are Kummer surfaces of a very special type: they are a quotient of an abelian surface which itself is 2-isogenous to $E \times E$ the self product of an elliptic curve. Then they use results for Kummer surfaces obtained by the last two mentioned authors in [**122**]. Finally they also use the lucky special situation to get such explicit bounds.

We give a summary of the criteria of Ieronymou, Skorobogatov and Zarhin from [**69**] and [**70**], since we use them in the next chapter.

PROPOSITION 3.146. *Let X be a diagonal quartic over $\mathbb{Q}$ which is given by the polynomial $a_0x_0^4 + a_1x_1^4 + a_2x_2^4 + a_3x_3^4$. Denote by M the subgroup in the multiplicative group of the rationals modulo forth powers $\mathbb{Q}^*/\mathbb{Q}^{(4)}$ generated by the ratios a_i/a_j for $i, j \in \{0, 1, 2, 3\}$ and -4. Let M' be the subgroup generated by M and -1. Then*

(1) $(\operatorname{Br}(X)/\operatorname{Br}_1(X))[2] \cong 0$ *if* $2 \notin M$,
(2) $(\operatorname{Br}(X)/\operatorname{Br}_1(X))[3] \cong 0$ *if* $3 \notin M'$,
(3) $(\operatorname{Br}(X)/\operatorname{Br}_1(X))[5] \cong 0$ *if* $5 \notin M'$.

For any positive prime number $p \geq 7$ we always have $(\operatorname{Br}(X)/\operatorname{Br}_1(X))[p] \cong 0$.

PROOF. For the statement about 2-torsion see [**69**, Theorem 5.2.], and for the other statements see [**70**, Theorem 3.2.]. □

CHAPTER 4

Examples

In this chapter we give an application of the theory recalled in the previous chapters to the example equation $x_0^4 + 3x_1^4 - 4x_2^4 - 9x_3^4 = 0$ and some related topics.

4.1. A Nontrivial Transcendental 3-Torsion Brauer Element

In this chapter we use the term "solutions of polynomial systems" and mean solutions to the system of equations associated by equating the polynomials to 0.

We look at the diagonal quartic surface X over the rationals $\mathbb{Q}$ defined by $x_0^4 + 3x_1^4 - 4x_2^4 - 9x_3^4$ in $\mathbb{P}^3_{\mathbb{Q}}$.

A straight forward computer search, e.g., computing $x_0^4 + 3x_1^4$ for a bounded set of pairs of integers (x_0, x_1), storing the results in a list, and comparing these values with the results of $4x_2^4 + 9x_3^4$ for a bounded set of pairs of integers (x_2, x_3), yields the following list of small primitive solutions:

```
[ 1, 1, 1, 0 ],
[ 7, 3, 5, 2 ],
[ 95, 63, 73, 36 ],
[ 43, 167, 155, 42 ],
[ 101, 201, 157, 130 ],
[ 331, 393, 103, 310 ],
[ 715, 39, 307, 398 ],
[ 299, 581, 65, 444 ],
[ 767, 273, 85, 448 ].
```

An integral solution is called primitive, if the greatest common divisor of the coordinates is 1. The list contains all primitive solutions satisfying $x_0^4 + 3x_1^4 \leq (1+3) \cdot (1000)^4$. In particular it contains all solutions with each coordinate below 1000. The quadruples are ordered according to the value of $x_0^4 + 3x_1^4$. See appendix C for a list of solutions satisfying $x_0^4 + 3x_1^4 \leq (1+3) \cdot (500000)^4$.

One can improve the simple search using congruences modulo small prime powers for the two terms $x_0^4 + 3x_1^4$ and $4x_2^4 + 9x_3^4$. However computer science aspects seem to be more important than mathematical ones. E.g., it is important for fast computations to avoid generating large lists of candidates, and checking on them later, but rather do the checking before appending them to a list. One is also forced to regroup the region which is searched into smaller partitions to avoid running out of memory. We did not spend to much effort on these issues. See the

Habilitation of Jahnel [**73**] or a recent preprint of Elsenhans [**37**] for a discussion of such matters.

Clearly having rational solutions implies els, so we do not have to worry about this condition in the further analysis. A careful inspection of this list yields that any integral solution $[y_0 : y_1 : y_2 : y_3]$ satisfies $3|y_1y_3$, but clearly there are 3-adic points on X which do not have this property, e.g.,

$$(\sqrt[4]{10}, 1, 1, 1)_{\mathbb{Q}_3}.$$

This is an indication but not a proof for failure of weak approximation.

THEOREM 4.1. *Weak approximation fails for X and every integral solution $[y_0 : y_1 : y_2 : y_3]$ to its definig equation $x_0^4 + 3x_1^4 - 4x_2^4 - 9x_3^4 = 0$ obeys the congruence $3|y_1y_3$. This failure is explained by a transcendental 3-trosion element of $\mathrm{Br}(X)$.*

Until the end of the proof, which occupies the whole section, we call it potential obstruction.

For the algebraic part of the Brauer group, we can use Bright's `MAGMA`-script, which is available at `www.boojum.org.uk/maths/quartic-surfaces/index.html`. Unfortunately, these useful script do not seem to run with newer versions of `MAGMA`, but they work at least with `MAGMA` version `V2.11-13`. The result is that

$$\mathrm{Br}_1(X)/\,\mathrm{Br}_0(X) \cong H^1_{gal}(\mathbb{Q}, \mathrm{Pic}(\overline{X})) \cong 0.$$

Alternatively we can read this off from the appendix of [**13**] as X can be defined by $(2x_2)^4 - 12x_1^4 - 4x_0^4 + (4 \cdot (-12)^2)(x_3/2)^4$ and thus falls into case `A282` but non of its subcases. This means any Brauer-Manin obstruction to weak approximation must come from a transcendental element of $\mathrm{Br}(X)$.

For the criterion of Ieronymou, Skorobogatov and Zarhin stated in proposition 3.146, we look at a group M' associated to X. Generators for M' are $-1, 4$ and the ratios of the coefficients, e.g., $3, -4, -9$ suffice, which yields a subgroup of $\mathbb{Q}^*/\mathbb{Q}^{(4)}$ generated by $-1, 4, 3$. This group does neither contain the class of 2, nor that of 5. Therefore, by the criterion of proposition 3.146, we have $(\mathrm{Br}(X)/\,\mathrm{Br}_1(X))[3] \cong \mathrm{Br}(X)/\,\mathrm{Br}_0(X)$, and any Brauer-Manin obstruction to explain this potential failure of WA must come from a transcendental 3-torsion element.

We need to construct transcendental 3-torsion elements in the Brauer group. To this end we use a fibration as proposition 3.127 suggests. After base change to the algebraic closure we have several candidate fibrations at our hand, namely the ones in proposition 2.114. The second of them was used by Ieronymou in [**69**] in computing 2-torsion in the transcendental Brauer group.

At first we tried this fibration, but did not succeed since computations got to complicated and time consuming even with `MAGMA`. After discussion with Ieronymou, who also tries to compute 3-torsion with this fibration and encountered similar problems we abandoned this approach. An account of the difficulties is given in section 4.4.

We then tried the third fibration in proposition 2.114, or rather a variant of it. Since the fibers are curves of geometric genus 3, the computation in the relative

Jacobian would be tremendously time consuming in general, but can in principle be done by effective Jacobian arithmetic (see [**66**], [**83**], [**84**] and [**14**]). In our special situation the complexity problems can be overcome with a finite morphism to a product of elliptic curves as discussed below.

Recently Okonek suggested to look at the first fibration instead of the third, but we were not able to do some work in this direction yet.

4.1.1. Overview of the Construction

The fibration f that we use is given by

$$\begin{tikzcd}
{[x_0 : x_1 : x_2 : x_3]} \arrow[d, mapsto] & \{x_0^4 + 3x_1^4 - 4x_2^4 - 9x_3^4 = 0\} =: X \arrow[d, dashed, "f"] & \subset \mathbb{P}^3_{\mathbb{Q}} \\
{[x_0 : x_1]} & \mathbb{P}^1_{\mathbb{Q}} &
\end{tikzcd}$$

The rational map f has a nontrivial locus of indeterminancy, which is given by $x_0 = x_1 = 0$ on X and consists of the single closed schematic point over $\mathbb{Q}$ given by $[0 : 0 : \zeta_8\xi : \sqrt[2]{2/3}]$ where $\xi \in \mu_4$ the set of forth roots of unity, and ζ_8 is an 8-th root of unity in $\mathbb{Q}^{alg}$. This point is not geometrically irreducible, but that is not a problem, e.g., in the following blowup operation.

Next we outline how to shift the computation of Brauer group elements from X to a related surface which is more accessible to calculations. Afterwards we explicitly carry this out on a dense open subscheme of X. For a clearer exposition we do not define many of the involved objects yet, but do this further on.

Consider the commutative diagram

$$\begin{tikzcd}
 & & & & & & X''' \arrow[dl, hook] \arrow[dr, hook] \arrow[drrr, "h'''"] & & & & \\
X \arrow[ddr, dashed, "f"'] & \tilde{X}' \arrow[l, "bl"'] \arrow[dd, "\tilde{f}"] & & \tilde{X}'_L \arrow[ll, "\tilde{h}"'] \arrow[dd, "\tilde{f}'"] & & \tilde{X}'' \arrow[ll, "\tilde{h}'"'] \arrow[dd, "\tilde{f}''"] \arrow[rrrrr, dashed, bend left] & & \tilde{X}''' \arrow[ll, "bl'"'] \arrow[rr, "\tilde{h}'''"] \arrow[dd, "\tilde{f}'''"] \arrow[rrr, bend left] & & \tilde{E} \times F_0 \arrow[r] \arrow[ddll, bend left] & F_0 \\
 & & X' \arrow[ul, hook] \arrow[dd] & & X'_L \arrow[ll, "h"'] \arrow[ul, hook, "\iota"'] \arrow[dd, "f'"] & & X'' \arrow[ll, "h'"'] \arrow[ul, hook] \arrow[uu, hook, "\iota'"'] \arrow[dd, "f''"] \arrow[rr, equal] & & X'' \arrow[ul, hook] \arrow[rr] \arrow[dd, "f''"] & & E \times F_0 \arrow[u] \arrow[ul, hook] \arrow[ddll, bend left] \\
 & \mathbb{P}^1_{\mathbb{Q}} & & \mathbb{P}^1_L \arrow[ll] & & \tilde{E} \arrow[ll, "\tilde{b}"'] \arrow[rr, equal] & & \tilde{E} & & & \\
 & & A \arrow[ul, hook] & & A_L \arrow[ll] \arrow[ul, hook] & & E \arrow[ll, "b"'] \arrow[ul, hook] \arrow[rr, equal] & & E \arrow[ul, hook] & &
\end{tikzcd}$$

Since the statements about blowups, desingularizations, loci of indeterminacy and generic degrees are straightforward calculations (see, e.g., [**61**, I.5. and III. Example 7.17.3.] and [**43**]), which involve only a few pages once the defining equations have been established (as we do below), we do not carry them out explicitly.

We concentrate on the $\tilde{\cdot}$-part. First by blowing up the locus of indeterminacy of f (see [**61**, II. Example 7.17.3.]) via bl, we get a smooth projective variety $\tilde{X}'$ and a morphism from $\tilde{X}'$ to $\mathbb{P}^1_{\mathbb{Q}}$, which is a fibration. Point blowups are proper birational morphism, and we can therefore apply proposition 3.78(1) to get $\mathrm{Br}(X) \cong \mathrm{Br}(\tilde{X}')$.

The blowup is performed working on the three affine charts for $x_0 = 1$, $x_1 = 1$ and $x_2 = 1$. The first two charts are not affected at all the blowup of the third can be described by two affine charts. We specify them by a set of defining polynomials and indicate the morphism to $\mathbb{P}^1_{\mathbb{Q}}$.

$$\begin{aligned}
\mathbb{A}^3_{\mathbb{Q}} \times_{\operatorname{Spec}\mathbb{Q}} \mathbb{A}^1_{\mathbb{Q}} &\supset \{1 + 3x_1^4 - 4x_2^4 - 9x_3^4, x_1 - t\} \to \mathbb{P}^1_{\mathbb{Q}}, (x_1, x_2, x_3, t) \mapsto [1 : t],\\
\mathbb{A}^3_{\mathbb{Q}} \times_{\operatorname{Spec}\mathbb{Q}} \mathbb{A}^1_{\mathbb{Q}} &\supset \{x_0^4 + 3 - 4x_2^4 - 9x_3^4, x_0 - s\} \to \mathbb{P}^1_{\mathbb{Q}}, (x_1, x_2, x_3, s) \mapsto [s : 1],\\
\mathbb{A}^4_{\mathbb{Q}} \times_{\operatorname{Spec}\mathbb{Q}} \mathbb{A}^1_{\mathbb{Q}} &\supset \{x_0^4 + 3x_1^4 - 4 - 9x_3^4, y_1 - t, x_0y_1 - x_1\} \to \mathbb{P}^1_{\mathbb{Q}}, (x_1, x_2, x_3, y_1, t) \mapsto [1 : t],\\
\mathbb{A}^4_{\mathbb{Q}} \times_{\operatorname{Spec}\mathbb{Q}} \mathbb{A}^1_{\mathbb{Q}} &\supset \{x_0^4 + 3x_1^4 - 4 - 9x_3^4, y_0 - s, x_1y_0 - x_0\} \to \mathbb{P}^1_{\mathbb{Q}}, (x_1, x_2, x_3, y_0, s) \mapsto [s : 1].
\end{aligned}$$

Then we perform two base extensions. The first is given by a Galois field extension $L/\mathbb{Q}$ of degree 16. We give a definition of L below where it will become clear from the context why we perform it.

Then we extend $\mathbb{P}^1_L$ to an elliptic curve $\tilde{E}$ by a degree 2 cover $\tilde{b}$ ramified in 4 points. $\tilde{E}$ is defined as a variety on weighted projective space, and $\tilde{X}''$ is defined via charts and has an obvious morphism to $\tilde{E}$, which is given via two affine charts.

$$\begin{gathered}
\mathbb{P}_L(1, 1, 2) \supset \tilde{E} := \{s^4 + 3t^4 - u^2\} \to \mathbb{P}^1_L, [s : t : u] \mapsto [s : t],\\
\tilde{E}_1 := \tilde{E} \setminus \{s = 0\}, \tilde{E}_2 := \tilde{E} \setminus \{t = 0\}\\
\mathbb{A}^3_L \times_{\operatorname{Spec} L} \tilde{E}_1 \supset \{1 + 3x_1^4 - 4x_2^4 - 9x_3^4, x_1 - t, 1 + 3t^4 - u^2\} =: \tilde{X}''_1,\\
\mathbb{A}^3_L \times_{\operatorname{Spec} L} \tilde{E}_2 \supset \{x_0^4 + 3 - 4x_2^4 - 9x_3^4, x_0 - s, s^4 + 3 - u^2\},\\
\mathbb{A}^4_L \times_{\operatorname{Spec} L} \tilde{E}_1 \supset \{x_0^4 + 3x_1^4 - 4 - 9x_3^4, x_0y_1 - x_1, y_1 - t, 1 + 3t^4 - u^2\},\\
\mathbb{A}^4_L \times_{\operatorname{Spec} L} \tilde{E}_2 \supset \{x_0^4 + 3x_1^4 - 4 - 9x_3^4, x_1y_0 - x_0, y_0 - s, s^4 + 3 - u^2\}.
\end{gathered}$$

$\tilde{X}''$ is not smooth. It has geometrically 4 point singularities. In the chart $\tilde{X}''_1$ they are given by the polynomials $\{x_2, x_3, u\}$ and have coordinates $((\xi\sqrt[4]{-1/3}, 0, 0), (\xi\sqrt[4]{-1/3}, 0))$, where $\xi \in \mu_4$. As $\tilde{h}'$ is a ramified morphism from a non-smooth variety to a smooth one it cannot be Galois. We cannot say anything about the behavior of $\mathrm{Br}(\tilde{h}')$ without analysing the ramification. We defer this until later.

We also have a rational map to another elliptic curve F_0 as follows:

$$\mathbb{P}_L(2,1,1) \supset F_0 := \{v^2 - 4y^4 - 9z^4\}$$
$$\{1 + 3x_1^4 - 4x_2^4 - 9x_3^4, x_1 - t, 1 + 3t^4 - u^2\} \dashrightarrow F_0, ((x_1, x_2, x_3), (t, u)) \mapsto [u : x_2 : x_3],$$
$$\{x_0^4 + 3 - 4x_2^4 - 9x_3^4, x_0 - s, s^4 + 3 - u^2\} \dashrightarrow F_0, ((x_0, x_2, x_3), (s, u)) \mapsto [u : x_2 : x_3],$$
$$\{x_0^4 + 3x_1^4 - 4 - 9x_3^4, x_0y_1 - x_1, y_1 - t, 1 + 3t^4 - u^2\} \to F_0, ((x_0, x_1, x_3, y_1), (t, u)) \mapsto [ux_0^2 : x_0 : x_3],$$
$$\{x_0^4 + 3x_1^4 - 4 - 9x_3^4, x_1y_0 - x_0, y_0 - s, s^4 + 3 - u^2\} \to F_0, ((x_0, x_1, x_3, y_0), (s, u)) \mapsto [ux_1^2 : x_1 : x_3].$$

Its locus of indeterminacy coincides exactly with its singularities. By blowing up via bl' we can remove the singular locus and get a smooth projective variety $\tilde{X}'''$ with a morphism to F_0. Since the rational map restricts to a morphism on the later two charts anyway and the blowup behaves essentially the same on the first two, we give the equations for the blowup only on $\tilde{X}_1''$. We get three charts

$$\mathbb{A}_L^3 \times_{\operatorname{Spec} L} \tilde{E}_1 \times_{\operatorname{Spec} L} \mathbb{A}_L^3 \supset \{1 + 3x_1^4 - 4x_2^4 - 9x_3^4, x_1 - t, 1 + 3t^4 - u^2, z_2u - x_2, z_3u - x_3, uu' - 1\} \to F_0,$$
$$((x_1, x_2, x_3), (t, u), (z_2, z_3, u')) \mapsto [u' : z_2 : z_3],$$
$$\mathbb{A}_L^3 \times_{\operatorname{Spec} L} \tilde{E}_1 \times_{\operatorname{Spec} L} \mathbb{A}_L^3 \supset \{1 + 3x_1^4 - 4x_2^4 - 9x_3^4, x_1 - t, 1 + 3t^4 - u^2, z_2x_3 - x_2, z_ux_3 - u, x_3x_3' - 1\} \to F_0,$$
$$((x_1, x_2, x_3), (t, u), (z_2, z_u, x_2')) \mapsto [z_u : z_2 : x_3'],$$
$$\mathbb{A}_L^3 \times_{\operatorname{Spec} L} \tilde{E}_1 \times_{\operatorname{Spec} L} \mathbb{A}_L^3 \supset \{1 + 3x_1^4 - 4x_2^4 - 9x_3^4, x_1 - t, 1 + 3t^4 - u^2, z_3x_2 - x_3, z_ux_2 - u, x_2x_2' - 1\} \to F_0,$$
$$((x_1, x_2, x_3), (t, u), (z_3, z_u, x_3')) \mapsto [z_u : x_2' : z_3].$$

This induces a morphism $\tilde{h}''' : \tilde{X}''' \to \tilde{E} \times_{\operatorname{Spec} L} F_0$, which turns out to be generally finite of degree 2. The locus where the morphism is not finite is the exceptional locus of the blowup bl'.

We then construct an explicit 2-cocycle $(q_1, q_2)_3$ representing a nontrivial element $\mathcal{A} \in \operatorname{Br}(\tilde{E} \times F_0)[3]$. We need to transfer its class to a class on $\operatorname{Br}(X)$. Unfortunately $\tilde{X}''$ and $\tilde{h}''$ are not nice enough for a direct transfer.

We have to consider open subvarieties missing the ramification loci and the loci of indeterminacy to transfer $\mathcal{A}$ to a potentially ramified Brauer class on $\tilde{X}_L'$ and deal with the ramification separately. To this end we introduce the un-$\tilde{\cdot}$-entries in the above big diagram.

Let $A := \mathbb{P}_{\mathbb{Q}}^1 \setminus \{[1 : \xi\sqrt[4]{-1/3}] : \xi \in \mu_4\}$ be the projective line without 4 points. After extension to L we see that A_L exactly misses the ramification locus of $\tilde{b}$. We now get an induced unramified extension $b : E \to A_L$ of degree 2, which is Galois. Induced from all this we get an open subvariety $\iota : X_L' \hookrightarrow \tilde{X}_L'$ missing the 4 singular fibers each of which are composed of 4 lines meeting in a point. $\mathcal{R}_{1,\tilde{X}_L' \setminus X_L'}$ therefore consists of 16 points which are the generic points of a rational curve. We also get a subvariety $X'' \hookrightarrow \tilde{X}''$, which misses also 16 rational curves making up the singular fibers of $\tilde{f}''$.

X_L' is smooth as a subvariety of a smooth variety, and X'' is smooth as a subvariety which misses the singular points that were contained in the removed fibers. Both are therefore regular (see remark 2.8). We have an induced morphism $h' : X'' \to$

X'_L, which is finite, since $\tilde{b}$ and therefore b is, unramified, because the ramification locus of $\tilde{h}'$ was removed, and flat by [**55**, IV. (6.1.5)], since X'_L is regular, X'' is regular hence Cohen-Macaulay and the relative dimension is constantly 0. We have a fixed point free involution on X'' over X'_L given in coordinates by $u \mapsto -u$ which shows that h' is an $H := \mathbb{Z}/2\mathbb{Z}$-Galois cover. Hence we can apply proposition 3.85 to get an isomorphism $h'^*[3]' : \mathrm{Br}(X'_L)[3] \to \mathrm{Br}(X'')[3]^H$.

By removing the 4 points of indeterminacy for the rational map $\tilde{X}'' \dashrightarrow F_0$ from $\tilde{X}''$ which coincides with its singular locus we get a smooth variety X''' which admits an obvious morphism $h''' : X''' \to \tilde{E} \times_{\mathrm{Spec}(L)} F_0$ and an embedding $\iota' : X'' \hookrightarrow X'''$. $\mathcal{R}_{1,X'''\setminus X''}$ consists of the generic points of the inverse images of the 16 lines which are represented by the generic points in $\mathcal{R}_{1,\tilde{X}'_L\setminus X'_L}$. The ramification locus of the induced morphism $X''' \to \tilde{X}'_L$ is made up exactly by the closure of $\mathcal{R}_{1,\tilde{X}'_L\setminus X'_L}$, and the ramification index is 2 for all 16 components.

Using remark 3.87 for the inclusions ι and ι', functoriality and a standard result on the behavior of the residue map under ramification (remark 3.72) we get the following commutative diagram with exact rows where m_2 is the multiplication by 2 map

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}(\tilde{X}'_L)[3] \arrow[r, "\mathrm{Br}(\iota)"] & \mathrm{Br}(X'_L)[3] \arrow[r, "\mathrm{ram}"] \arrow[d, "\mathrm{Br}(h')"] \arrow[dr, phantom, "\circlearrowleft"] & \bigoplus_{y\in\mathcal{R}_{1,\tilde{X}'_L\setminus X'_L}} H^1_{gal}(\kappa_y,\mu_3) =: \mathbf{X}_1 \arrow[d, "m_2"] \\
0 \arrow[r] & \mathrm{Br}(X''')[3] \arrow[r, "\mathrm{Br}(\iota')"] & \mathrm{Br}(X'')[3] \arrow[r, "\mathrm{ram}'"] & \bigoplus_{y\in\mathcal{R}_{1,X'''\setminus X''}} H^1_{gal}(\kappa_y,\mu_3) =: \mathbf{X}_2 \\
& \mathrm{Br}(\tilde{E}\times_{\mathrm{Spec}(L)} F_0). \arrow[u, "\mathrm{Br}(h''')"] & &
\end{tikzcd}$$

Our construction of $(q_1, q_2)_3$ will not only ensure that $(\mathrm{Br}(\iota') \circ \mathrm{Br}(h'''))(\mathcal{A})$ is H-invariant, but that even its representative, which is essentially $(q_1, q_2)_3$ itself, is H-invariant. Therefore there is a class $\mathcal{B} \in \mathrm{Br}(X'_L)$ satisfying $(\mathrm{Br}(\iota') \circ \mathrm{Br}(h'''))(\mathcal{A}) = \mathrm{Br}(h')(\mathcal{B})$. $\mathcal{B}$ is in $\mathrm{im}(\mathrm{Br}(\iota))$, if and only if its ramification is trivial, i.e., $\mathrm{ram}(\mathcal{B}) = 0$. If it was not, it had to be an element of order 3 in $\mathbf{X}_1$ and would be mapped to a non-vanishing element in $\mathbf{X}_2$ by m_2. This contradicts commutativity of the square and exactness of the lower row. Thus $(q_1, q_2)_3$ represents an unramified Brauer element $\mathcal{C} \in \mathrm{Br}(\tilde{X}'_L)$.

$\mathcal{C}$ can be interpreted as an unramified element of $\mathrm{Br}(k(\tilde{X}'_L))$. Using the compatibility of corestriction and the ramification map of remark 3.72 we get an unramified element $\mathcal{D} \in \mathrm{Br}(k(\tilde{X}'))$ which by proposition 3.75 gives an element of $\mathrm{Br}(\tilde{X}')$. By the birationality of a blowup we have $\mathrm{Br}(X) \cong \mathrm{Br}(\tilde{X}')$ and finally a Brauer class defined over X. The corestricted Brauer class might be constant, but our computations below show that it is not.

4.1.2. Motivation for the Construction

Here is the computation for the chosen model, which gives a motivation why the above diagram is promising to give an obstruction. Denote $U := 1+3t^4$ and $A_* := \mathbb{A}^1_{\mathbb{Q}} \setminus \{(\xi\sqrt[4]{-1/3}) : \xi \in \mu_4\}$. We take the following model $X'_* \subset A_* \times_{\mathrm{Spec}(\mathbb{Q})} \mathbb{P}^2_{\mathbb{Q}}$, which avoids the singular fibers and an additional fiber at ∞. We have $X'_* \subset X' \subset \tilde{X}'$, and X'_* is obtained by gluing those two of the four charts for $\tilde{X}'$ which involve the parameter t.

$$
\begin{tikzcd}
{[x : tx : y : z]} & {(t, [x : y : z])} \arrow[l, mapsto] \\
\{x_0^4 + 3x_1^4 - 4x_2^4 - 9x_3^4 = 0\} \arrow[d, dashed, "f"] & \left\{\begin{matrix} Ux^4 - 4y^4 - 9z^4 = 0 \\ U = 1 + 3t^4 \neq 0 \end{matrix}\right\} =: X'_* \arrow[l, hook'] \arrow[d, "f'"] & (t, [x : y : z]) \arrow[d, mapsto] \\
\mathbb{P}^1_{\mathbb{Q}} & A_* \arrow[l, hook'] & (t).
\end{tikzcd}
$$

From now on the condition $U \neq 0$ is implicit for all calculations in coordinates. We omit it for simplicity. The calculations will be carried out with the $\bullet_*$-versions and we interpret the results in terms of the non-$_*$-variants.

By the analogous version of propositions 3.86 for $H^1(\bullet, R^1\pi_*\mathbb{G}_m)$, proposition 3.127, and definition 3.120 we have:

$$
H^1_{\acute{e}t}(\overline{A}, \underline{\mathrm{Pic}}_{\overline{X'}/\overline{A}}[3]) \hookleftarrow H^1_{\acute{e}t}(\overline{\mathbb{P}^1}, \underline{\mathrm{Pic}}_{\overline{\tilde{X}'}/\overline{\mathbb{P}^1}}[3]) \twoheadrightarrow \mathrm{Br}(\overline{\tilde{X}'})[3] \cong \mathrm{Br}(\overline{X})[3] \hookleftarrow (\mathrm{Br}(X)/\,\mathrm{Br}_1(X))[3].
$$

This is the starting point for our interest in f.

$\underline{\mathrm{Pic}}_{\overline{X'}/\overline{A}}[3]$ is a subsheaf of $\underline{\mathrm{Pic}}^0_{\overline{X'}/\overline{A}}$, which is representable by a scheme, namely the relative Jacobian fibration $\mathrm{Jac}_{\overline{X'}/\overline{A}}$ (see, e.g., [**18**, 8.2. Theorem 1 and 9.3. Theorem 1], and take into account that we removed all geometrically non-irreducible fibers). Since each fiber is defined by an equation $\{(1+3t_0^4)x^4 - 4y^4 - 9z^4 = 0\} \subset \mathbb{P}^2_{\mathbb{Q}(1+3t_0^4)}$ which is always a smooth plane quartic (since the non-smooth fibers were removed), the Jacobian fibration is a fibration in abelian 3-folds, which is very hard to compute in general. However, we have the following degree 2 maps into varieties supposed to be embedded into a relative weighted projective space $A_* \times_{\mathrm{Spec}(\mathbb{Q})} \mathbb{P}_{\mathbb{Q}}(2,1,1)$, where the weighted coordinate varies for each map:

$$
\begin{tikzcd}
 & \{Ux^4 - 4y^4 - 9z^4 = 0\} = X'_* \arrow[dl, "p_0"'] \arrow[d, "p_1"] \arrow[dr, "p_2"] & \\
\{Ux^2 - 4y^4 - 9z^4 = 0\} \arrow[dr] & \{Ux^4 - 4y^2 - 9z^4 = 0\} \arrow[d] & \{Ux^4 - 4y^4 - 9z^2 = 0\} \arrow[dl] \\
 & A &
\end{tikzcd}
$$

Call the new fibered surfaces $Y_0, Y_1, Y_2 \subset A_* \times_{\mathrm{Spec}(\mathbb{Q})} \mathbb{P}_{\mathbb{Q}}(2,1,1)$. All the Y_i avoid the quotient singularity of the weighted projective space, e.g., the singularity $(t_0, [1:0:0])$ does not lay on Y_0, and similarly for the other Y_i.

We can look at the new surfaces as relative double covers of the relative projective line and thus as elliptically fibered surfaces. They are relative elliptic curves admitting sections over A_* given by $x = 0$. By the universal property of Jacobians we get morphisms $\mathrm{Jac}_{\overline{X'_*}/\overline{A_*}} \xrightarrow{p'_i} \overline{Y_i}$ induced by the p_i. By the universal property of the product they in turn induce

$$\mathrm{Jac}_{\overline{X'_*}/\overline{A_*}} \xrightarrow{p' := p'_0 \times p'_1 \times p'_2} \prod_i \overline{Y_i}.$$

Coincidentally, all geometric fibers of any Y_i are isomorphic elliptic curves of j-invariant 1728, which turns out to be important for practical computations. We find that p' is a finite morphism of degree $8 = 2^3$. The computations can be found in section 4.2.

We compute $H^1_{\acute{e}t}(\overline{A}, \underline{\mathrm{Pic}}_{\overline{X'}/\overline{A}}[3])$. Remember corollary 3.51 and take into account that $\underline{\mathrm{Pic}}_{\overline{X'}/\overline{A}}[3]$ is a sheaf of finite abelian groups:

$$H^1_{grp}(\widehat{\pi_1^{an}(A^{an})}, \mathrm{Pic}_{X'/A}[3]) \cong H^1_{\acute{e}t}(\overline{A}, \underline{\mathrm{Pic}}_{\overline{X'}/\overline{A}}[3]).$$

Since $\mathrm{Pic}_{X'/A}[3]$ is a finite group, the $\widehat{\pi_1^{an}(A^{an})}$-group action factors through a finite quotient, which by definition of the pro-finite completion is also a quotient of $\pi_1^{an}(A^{an})$. We thus may use the non-completed analytic fundamental group.

We may vary coefficients to our convenience, since we now work over $\mathbb{C}$, and use the fibration of the surface X'_* given by

$$A_{*\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{P}^2_{\mathbb{C}} \supset \{(t, [x,y,z]) : (1+t^4)x^4 - y^4 - z^4 = 0\} \to A_{*\mathbb{C}}.$$

We plan to apply the degree 8 finite morphism, which gives also a degree 8 morphism in the analytic category, and define three covers which are restrictions of

$$\begin{aligned}
A_{*\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{P}_{\mathbb{C}}(2,1,1) &\supset \{(t, [x,y,z]) : (1+t^4)x^2 - y^4 - z^4 = 0\} \to A_{*\mathbb{C}},\\
A_{*\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{P}_{\mathbb{C}}(1,2,1) &\supset \{(t, [x,y,z]) : (1+t^4)x^4 - y^2 - z^4 = 0\} \to A_{*\mathbb{C}},\\
A_{*\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{P}_{\mathbb{C}}(1,1,2) &\supset \{(t, [x,y,z]) : (1+t^4)x^4 - y^4 - z^2 = 0\} \to A_{*\mathbb{C}},
\end{aligned}$$

of which the last two can be treated identically. We denote them by $F_i \to A_{*\mathbb{C}}$. Computation in weighted projective spaces is not implemented in `MAGMA`, so we resort to the trick of using another degree 2 cover for each F_i to their relative Jacobians in projective space. In our case there are well-known explicit formulas for Jacobians of genus 1 curves and associated morphism, see, e.g., [**3**].

We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Jac}^{an}_{X'_*/A_*} \arrow[r, "\deg=8"] \arrow[dr] & (\prod_i F_i)^{an} \arrow[r, "\cong"] \arrow[d] & \prod_i F_i^{an} \arrow[r, "\deg=2^3"] \arrow[dl] & \prod_i(\mathrm{Jac}^{an}_{F_i/A}) \arrow[dll] \\
 & A^{an} & &
\end{tikzcd}$$

We use the splitting into a relative product, and proceed by computing $\mathrm{Jac}^{an}_{F_i/A_*}[3]$ and its $\pi_1^{an}(A_*^{an})$ group action using discrete numerical path following techniques. The various coverings might have trivialized some of the action (and they actually do), so again by path following techniques we lift the action from the Jacobian to the F_i-covers. These covers do not annihilate the 3-torsion itself only part of the action, since the degrees of the maps are powers of 2 and hence prime to 3.

While the cover of F_i to its Jacobian was too complicated to analyze the modification of the action by hand, the cover from X'_* to the F_i is given by squaring one coordinate. Inspection shows that these covers cannot make the action more complicated. The action on $\mathrm{Pic}_{X'_*/A_*} \cong (\mathbb{Z}/3\mathbb{Z})^6$ factors in three blocks corresponding to the three F_i.

We have $\pi_1^{an}(A^{an}) = \pi_1^{an}((\mathbb{P}^1_{\mathbb{C}} \setminus \{\xi\sqrt[4]{-1} : \xi \in \mu_4\})^{an})$ is isomorphic to the free group on 3 generators. $\pi_1^{an}(A_*^{an})$ would have an additional free generator. But keep in mind that we do calculations for A via the more convenient model A_*, so we need to consider only the image of the natural inclusion $\pi_1^{an}(A^{an}) \hookrightarrow \pi_1^{an}(A_*^{an})$. The action on the F_0-cover is identical for the three generators chosen for $\pi_1^{an}(A^{an})$. It factors through $\mathbb{Z}/2\mathbb{Z}$ and is fixed point free on the liftings of the nontrivial 3-torsion. Thus the action on the lifts of the 3-torsion on F_0 also factors through $\mathbb{Z}/2\mathbb{Z}$. Since it also has to give rise to a matrix representation on the 3-torsion subgroup isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$, we see that the action is simply given by multiplication by 2.

For the F_1- and F_2-cover we similarly get a factorization of the action through $\mathbb{Z}/4\mathbb{Z}$, such that the action of twice a generator is fixed point free on nontrivial lifts of 3-torsion. All three generators give the same action.

We now give a matrix representation of the action in terms of $\mathbb{Z}/3\mathbb{Z}$-matrices. For the F_0-part we have multiplication by 2 thus $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. For the F_1/F_2-part we get that twice a generator acts as multiplication by 2. Thus the action on a chosen basis vector has to be of the form $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \mapsto v \mapsto \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right)$. We choose $v := \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, and get $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$.

We get for each generator of $\pi_1^{an}(A^{an})$ the same action represented by

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

The numerical path lifting algorithms are not provably correct, for we did not compute error bounds and made sure that the numerical values stay inside that bounds. It is however very likely that the algorithms give the correct result. In the end we use the results from the numerical computation to get an idea which exact étale cover and which cocycle could be interesting. The construction of a cocycle itself does not depend on the numerical calculation, only the choice of it as a promising candidate.

With `MAGMA` we can compute the whole cohomology group $H^1_{grp}(\pi_1^{an}(A^{an}), \mathrm{Pic}_{X'/A}[3])$. However we are not really interested in the cohomology group itself, rather in nontrivial elements which admit small representations. For easier handling later on small representation are favourable. Using the Lyndon-Hochschild-Serre spectral sequence and its associated exact sequence in low degree terms, better known as inflation-restriction sequence, we determine a finite quotient of $\pi_1^{an}(A^{an})$ over which all of the cohomology is defined. Then we search for smaller quotients over which at least some nontrivial 3-torsion is defined. We discuss this separately in section 4.3, where we also give more details on determining the group action in general.

The result is that a degree 6 cover of A has good chances to host nontrivial 3-torsion elements in the form of étale cocycles.

We computed $H^1_{grp}(\widehat{\pi_1^{an}(A^{an})}, \mathrm{Pic}_{X'/A}[3])$, and from this we restricted to $H^1_{grp}(Q, M)$ for an appropriate quotient $Q \cong S_3$ of $\widehat{\pi_1^{an}(A^{an})}$, by the inflation-restriction sequence as in proposition 3.60. We aim for representatives of $H^1_{\acute{e}t}(\overline{A}, \underline{\mathrm{Pic}}_{\overline{X'}/\overline{A}}[3])$. Such representatives are (Čech) étale 1-cocycle, i.e., elements of $(\underline{\mathrm{Pic}}_{\overline{X'}/\overline{A}}[3])(\hat{A} \times_{\overline{A}} \hat{A})$ for a Galois étale cover $\hat{A} \to \overline{A}$. In our situation it suffices to look at a single étale cover instead of surjective families.

This can be done by effective Riemann existence theorem. However this in general involves constructing a suitable discrete "subspace of covers associated to a given Galois group and the given curve" as the intersection of several conditions in the moduli spaces of curves (see [**2**]). These results can be made effective with the help of effective invariant theory (see [**35**]). But it is not readily implemented and computationally probably very expensive.

The degree 6 étale cover encoded in the group cohomology cocycle we found splits into a degree 2 quotient and a degree 3 cover. The prime to 3 part does not contribute to a 3-torsion element, so we extend this part right away in taking the étale cover $\mathbb{P}_L(1,1,2) \supset \overline{E} \to \overline{A}$ which is given by introducing a new coordinate u satisfying $u^2 = s^4 + 3t^4$. The equation $u^2 = s^4 + 3t^4$ is read off from the group cohomology data.

The completion of $\overline{E}$ is an elliptic curve, since $u^2 = s^4 + 3t^4$ defines a double cover of $\mathbb{P}^1_{\mathbb{Q}^{alg}}$ ramified in 4 points, i.e., a hyperelliptic curve of genus 1, and $\overline{E}$ has rational points, e.g., $[s : t : u] = [1 : 0 : 1]$. We work with an affine model $\overline{E_*}$. $\overline{E_*}$ misses the following 6 points of the completion: 2 points at $s = 0$, i.e., at ∞, and 4 which form the ramification locus. We set $s = 1$.

We take the pullback of our model to the new base:

$$\begin{tikzcd}
\overline{X'_*} = \{(1+3t^4)x^4 - 4y^4 - 9z^4 = 0\} \arrow[d, "\overline{f'_*}"] \arrow[dr, phantom, "\llcorner", very near start] & \{u^2x^4 - 4y^4 - 9z^4 = 0\} = \overline{X''_*} \arrow[l] \arrow[d, "\overline{f''_*}"] \\
\overline{A_*} & \overline{E_*}. \arrow[l, "u^2=1+3t^4"']
\end{tikzcd}$$

The splitting of the group cohomology cocycle in a prime to 3 and a 3-power part works, because we are in the simple case $Q \cong S_3$ and the 3-Sylow subgroup is normal. This is not possible in general.

Now we need to find a Galois étale 3-cover $\overline{\hat{E}} \to \overline{E}$, necessarily cyclic, and a nontrivial representative in $(\underline{\mathrm{Pic}}_{\overline{X''}/\overline{E}}[3])(\overline{\hat{E}} \times_{\overline{E}} \overline{\hat{E}})$.

Instead of reading of $\overline{\hat{E}}/\overline{E}$ from the group cohomology cocycle, we test all possible covers. This is just more convenient. For $\overline{E}$ the possible covers of degree 3 which are étale, hence unramified, are covers of its completion $\overline{\tilde{E}}$ ramifying at the missing points, or isogenies of degree 3 of $\overline{\tilde{E}}$ restricted to $\overline{E}$. Any other covers can be excluded by the Riemann-Hurwitz theorem (see [**61**, IV. Corollary 2.4.]), which gives bounds on the genus of curves depending on the prescribed ramification.

We only consider isogenies, since we want an unramified and nontrivial Brauer group element of X and for this we need unramified covers.

Isogenies of elliptic curves are in 1 : 1-correspondence to endomorphisms of 1-dimensional complex lattices by GAGA, the Riemann-existence theorem and the fact that the complex plane modulo an embedded lattice give abelian curves, i.e., elliptic curves. Such endomorphisms of degree 3 correspond to the 3 torsion points modulo that lattice, hence to the 3-torsion of the Jacobian of the elliptic curve which may be identified with the 3-torsion in the elliptic curve itself.

Now we describe the coefficients. $\underline{\mathrm{Pic}}_{\overline{X''}/\overline{E}}[3]$ can be represented by a subscheme of the Jacobian $\mathrm{Jac}_{\overline{X''}/\overline{E}}$, i.e, a relative abelian 3-fold. We see in section 4.2 that this Jacobian admits a degree 8 morphism to a product of relative genus 1 curves. We apply a variant where $v = ux^2$:

$$
\begin{tikzcd}
 & \{u^2x^4 - 4y^4 - 9z^4 = 0\} = \overline{X''_*} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{v^2 - 4y^4 - 9z^4 = 0\} \arrow[dr] & \{u^2x^4 - 4y^2 - 9z^4 = 0\} \arrow[d] & \{u^2x^4 - 4y^4 - 9z^2 = 0\} \arrow[dl] \\
 & \overline{E} &
\end{tikzcd}
$$

This modification makes use of the extension to $\overline{E}$ insofar that the first factor now varies trivially with the parameter of the base curve, which is exactly why we performed this extension $\overline{E/A}$ in the first place. We denote the factors by $\overline{F'_i}$. It is enough to work with factor $\overline{F'_0}$ and assume that the contribution of $\overline{F'_1}/\overline{F'_2}$ is trivial, since the found group cohomology cocycle only has nontrivial values in the block corresponding to $\overline{F'_0}$. Coincidentally $\overline{F'_0}$ is geometrically isomorphic to $\overline{E}$, but to prevent confusion we denote these curves by different names.

For the isomorphism $H^1_{grp}(\widehat{\pi_1^{an}(A^{an})}, \mathrm{Pic}_{X'/A}[3]) \cong H^1_{ét}(\overline{A}, \underline{\mathrm{Pic}}_{\overline{X'}/\overline{A}}[3])$ we had to base change to the algebraic closure. Now we changed to $H^1_{ét}(\overline{E}, \overline{F'_0}[3])$. In the light of remark 3.52 it suffices to base change to a finite field extension L with

$F'_{0L}[3] \cong (\mathbb{Z}/3\mathbb{Z})^2$. $\overline{F'_0}/\overline{E}$ is a constant family of elliptic curves. The 3-torsion therefore is defined over an algebraic extension of $\mathbb{Q}$

We define L to be the field of definition of any fiber $\overline{F'_0}[3]_a$ defined over $\mathbb{Q}$. We get $L := \mathbb{Q}(\zeta_{12}, \rho)$, where $\rho := \sqrt[4]{3 + 2\zeta_{12} - 3\zeta_{12}^2 - 4\zeta_{12}^3}$ and ζ_{12} is a 12-th root of unity. L is isomorphic to the splitting field of $x^{16} - 6x^{12} + 39x^8 + 18x^4 + 9$ and is a degree 16 Galois extension $L/\mathbb{Q}$.

Let $\tilde{E}$ be the degree 32 extension of $\mathbb{P}^1_{\mathbb{Q}}$ given by $L/\mathbb{Q}$ and $u^2 = s^4 + 3t^4$. Let E be the affine part consistent with the above $\overline{E}$. E misses only $2 + 2$ instead of $2 + 4$ points, since the ramification locus is not split by L. We descent $\overline{F'_0}/\overline{E}$ to F'_0/E defined over L, and similarly in analogous situations. It is easy to check that all conditions for this descent are met.

The morphism $X'' \to F'_0$ extends to a rational map $\tilde{X}'' \dashrightarrow \tilde{F}'_0$. Blowing up the locus of indeterminacy via bl' yields $\tilde{X}''' \to F'_0$.

To avoid potential ramification at codimension 1 points of deleted fibers over the base curve E, we look for 3-torsion elements in F'_0/E that are constant with respect to the base curve E, which is possible since F'_0 is a constant family of elliptic curves. We pick a fiber $F_0 := F'_{0,b}$ for some $b \in E$. We take a b such that $F_0[3]$ is defined over L. This is not essential but helpful, since we needed to work in a bigger field extension otherwise. Instead of computing in a relative elliptic curve F'_0 or more precisely in its function field $k(E) = L(t)[u = \sqrt{1 + 3t^4}]$, we simply compute on an elliptic curve over L. This reduces computational complexity drastically.

We convert the étale cocycle into a different representation. Since $\hat{E}/E$ is Galois of degree 3 and therefore has Galois group $G \cong \mathbb{Z}/3\mathbb{Z}$, we have $\hat{E} \times_E \hat{E} \cong \hat{E} \times_E G$. We can give the 1-cocycle as a cyclic algebra of degree 3, and this is specified by giving a pair of rational functions denoted as $(q_1, q_2)_3$, where the first specifies a G-field extension of the function field of degree 3 and the second determines the multiplicative behavior of the algebra generator over that field extensions (cf., e.g., [**103**, 15.1.]). This interpretation is a bit arbitrary, since the algebra associated to the pair with swaped entries represents the inverse associated sAa. This is reflected in that (q_1, q_2) represents an element of $\mathrm{Br}(E \times_{\mathrm{Spec}(L)} F_0) = \mathrm{Br}(F'_0)$.

We describe the conversion for E; it works analogously for F_0. Let $P - Q$ be a 3-torsion point in E respectively in its Jacobian. The associated field extension is given by the third root of the rational function q that proves the linear equivalence of $3(P - Q)$ to the 0-class, i.e.,

$$(q) = 3(P - Q).$$

This is the case because we want all torsion points in the subgroup generated by $P - Q$ to be annihilated by the cover $\hat{E}/E$, or equivalently to become principle, and this is the case when $\sqrt[3]{q}$ becomes a rational function, i.e.

$$\forall n \in \mathbb{Z} : (\sqrt[3]{q}^n) = n(P - Q).$$

q then is an element of $k(\tilde{E})$, but may be considered an element of $k(\tilde{E} \times_{\mathrm{Spec}(L)} F_0)$. There are also implementation issues to consider, e.g., at some point we passed to weighted projective space, but this is not supported out of the box by the computer algebra system we use. One has to be careful and not literally take the computed rational functions, but rather apply some transforms first.

We compute candidates for $(q_1, q_2)_3$ with `MAGMA`.

$$\begin{aligned}
q_1 := (&12(3\eta^{14} - 26\eta^{10} + 156\eta^6 - 219\eta^2)t^3 + \\
&24(-4\eta^{12} + 26\eta^8 - 156\eta^4 - 33)t^2 + 6(\eta^{15} + 213\eta^3)u + \\
&4(-25\eta^{14} + 156\eta^{10} - 1014\eta^6 - 333\eta^2)t - 936 + \\
&6(-33\eta^{15} - 8\eta^{13} + 195\eta^{11} + 39\eta^9 - 1287\eta^7 - 273\eta^5 - 828\eta^3 - 27\eta)ut^3 + \\
&6(79\eta^{15} - 494\eta^{11} + 3198\eta^7 + 681\eta^3)ut^2 + \\
&6(-17\eta^{15} - 2\eta^{13} + 117\eta^{11} + 13\eta^9 - 741\eta^7 - 39\eta^5 + 162\eta^3 - 75\eta)ut + \\
&3(20\eta^{14} - 31\eta^{12} - 117\eta^{10} + 182\eta^8 + 741\eta^6 - 1326\eta^4 + 711\eta^2 - 285)u^2t^2 + \\
&2(25\eta^{14} + 27\eta^{12} - 156\eta^{10} - 117\eta^8 + 1014\eta^6 + 1053\eta^4 + 333\eta^2 - 216)u^2t + \\
&9(-6\eta^{14} + \eta^{12} + 39\eta^{10} - 247\eta^6 + 9\eta^2 + 213)u^2 + \\
&(-3\eta^{15} + 37\eta^{13} - 195\eta^9 + 1443\eta^5 - 639\eta^3 + 666\eta)u^3)/ \\
&(1872 + 24(-\eta^{15} + 6\eta^{13} - 39\eta^9 + 234\eta^5 - 252\eta^3 + 108\eta)u + \\
&36(6\eta^{14} - 39\eta^{10} + 247\eta^6 - 9\eta^2 + 78)u^2 + \\
&2(-3\eta^{15} + 31\eta^{13} - 195\eta^9 + 1209\eta^5 - 873\eta^3 + 558\eta)u^3),
\end{aligned}$$

$$\begin{aligned}
q_2 := (&72(-7\eta^{12} + 52\eta^8 - 312\eta^4 + 69)y^3z^3 + \\
&936(\eta^{12} - 6\eta^8 + 42\eta^4 + 9)y^2z^4 + 108(-9\eta^{12} + 52\eta^8 - 312\eta^4 - 201)yz^5 - \\
&25272z^6 + 12(-5\eta^{15} - 27\eta^{13} + 195\eta^9 - 1287\eta^5 - 1377\eta^3 + 684\eta)ux^2y^3z + \\
&156(-\eta^{15} + 6\eta^{11} - 42\eta^7 + 9\eta^3)uy^2z^2 + \\
&18(-79\eta^{15} - 57\eta^{13} + 494\eta^{11} + 351\eta^9 - 3198\eta^7 - 2223\eta^5 - 681\eta^3 - 1026\eta)uyz^3 + \\
&18(53\eta^{15} - 312\eta^{11} + 2028\eta^7 + 1305\eta^3)uz^4 + \\
&2(54\eta^{14} - 85\eta^{12} - 351\eta^{10} + 546\eta^8 + 2223\eta^6 - 3354\eta^4 - 81\eta^2 - 711)u^2y^2 + \\
&6(-120\eta^{14} + 9\eta^{12} + 741\eta^{10} - 52\eta^8 - 4797\eta^6 + 312\eta^4 - 1575\eta^2 + 201)u^2yz + \\
&9(-12\eta^{14} - 3\eta^{12} + 65\eta^{10} - 429\eta^6 - 411\eta^2 - 171)u^2z^2 + \\
&(-53\eta^{15} - 43\eta^{13} + 312\eta^{11} + 221\eta^9 - 2028\eta^7 - 1365\eta^5 - 1305\eta^3 - 2490\eta)u^3)/ \\
&(108(-11\eta^{15} + 8\eta^{13} + 65\eta^{11} - 39\eta^9 - 429\eta^7 + 273\eta^5 - 276\eta^3 + 495\eta)uz^4 + \\
&50544z^6 + 36(12\eta^{14} - 65\eta^{10} + 429\eta^6 + 411\eta^2 + 234)u^2z^2 + \\
&2(20\eta^{15} + 11\eta^{13} - 117\eta^{11} - 52\eta^9 + 741\eta^7 + 390\eta^5 + 477\eta^3 + 705\eta)u^3).
\end{aligned}$$

The above are the rational functions on X''_* in coordinates $((t:u),[1:y:z])$ that we choose for q_1, q_2. Remember that L was the splitting field of $x^{16} - 6x^{12} + 39x^8 + 18x^4 + 9$ and we denote a root of this polynomial by η. Restricting to the chart

$x = 1$ will not be a problem as discussed below; in the MAGMA computations we have to replace u by $ux^2 = v$ in q_2. One can make choices regarding the 3-torsion point $P - Q$ – up to sign 4 choices for E and also for F_0 to be precise. Not all choices explain $3|x_1x_3$, but with the choice we state we succeed.

4.1.3. Ramification and Local Analysis

Finally we have to show that the constructed Brauer element representative $(q_1, q_2)_3$ comes from a nontrivial class in $\mathrm{Br}(X_L)$ and gives rise to an obstruction to WA.

This also implies that the constructed element actually is a non-constant transcendental 3-torsion element. For this we recall the ramification analysis above. The only thing that remains is to show that the class of $(q_1, q_2)_3$ is unramified on $\tilde{E} \times_{\mathrm{Spec}(L)} F_0$ giving rise to a Galois invariant class on X''.

Both functions q_1, q_2 were associated to 3 torsion points on their respective elliptic curve and had zeros and poles associated to some $3P - 3Q$. Let T be a nontrivial 2 torsion point and $\oplus$ the group law, then the rational function for the divisor $3(P \oplus T) - 3(Q \oplus T)$ gives rise to an equivalent Brauer element, but has poles and zeros on a disjoint locus on the factor curve compared to the previous poles and zeros. This is possible, since all 2-torsion points are defined over L.

According to [**78**, Proposition 5.(iv) and Example 8.] to get to descent the $\mathbb{Z}/3\mathbb{Z}$-data to a $S_3 \cong D_3$-data the rational function q_2 must be Galois anti-invariant. This can be achieved by a simple normalization because we constructed our function from a three torsion element $P - Q$ where P and Q are permuted among each other by the involution constituting the nontrivial Galois automorphism. Thus we get an unramified Brauer class over X. To show that it is non-constant we use the local computations at various places p.

To prove the obstruction to WA we only need to find a prime p and p-adic points P, P' on X such that evaluation of $(q_1, q_2)_3$ has different values in $\mathbb{Q}/\mathbb{Z}$ at P and P'. We prove a little more in showing that $3|x_1x_3$ for all integral solutions $[x_0 : x_1 : x_2 : x_3]$.

Now we give a discussion of the local behavior at the places of $\mathbb{Q}$. By remark 3.136 we need to check the places S' of bad reduction of X, places S'' of bad reduction of $[(q_1, q_2)_3]$ and real places $\Omega_{\mathbb{Q}}^{\mathbb{R}}$.

Our Azumaya algebra is given over L. However we are only interested to evaluate the class of $(q_1, q_2)_3$ at $\mathbb{Q}$-points (or points for the associated local field). Thus in the end we are only interested in the places of $\mathbb{Q}$ laying below the places where the invariants for L are not guaranteed to be constant. We also have to include the primes at which $L/\mathbb{Q}$ ramifies in the set S''.

For the non-archimedian places one definitely has to check 2 and 3, since they are the primes S' where X does not have a smooth model. MAGMA computes the discriminant to be

$$\mathrm{disc}(\mathfrak{o}_L) := 2^{32} \cdot 3^{14}.$$

Thus S'' includes the ramified primes 2 and 3.

In summary for a complete local analysis we would have to check at least $2, 3$ and the real place. The algebra $(q_1, q_2)_3$ is of good reduction at all places, where $\tilde{E}$ and F_0 reduce to elliptic curves with well-defined 3 torsion elements. This is essentially due to the fact, that $(q_1, q_2)_3$ represents a cup product of elements of $H^1_{ét}(\tilde{E}, \mu_3)$ and $H^1_{ét}(F_0, \mu_3)$ (remember that L contains 3rd roots of unity). The places of bad reduction for $\tilde{E}, F_0$ are $\{2, 3\}$ as inspection of the defining equations $s^4 + 3t^4 = u^2$ and $v^2 - 4y^4 - 9z^4 = 0$ shows. Thus the evaluation away from $2, 3$ and the real place is trivial.

We discuss the real place first, then 2, and finally 3.

The local invariant at a real place is either 0 or $1/2$. Since the order of the class of $(q_1, q_2)_3$ is either 1 or 3 and hence not divisible by 2, the invariant has to be 0.

Now for the 2-adic points. We need to evaluate the algebra $(q_1, q_2)_3$ given on the affine model X''_* at $\mathbb{Q}_2$-rational points on X. Let $[1 : x_1 : x_2 : x_3]$ be a 2-adic point on X; since it turns out that there are no solutions with $x_0 = 0$ this is no restriction. It corresponds to $([s : t : u], [x : y : z]) = ((1 : x_1 : \sqrt{1 + 3x_1^4}], [1 : x_2 : x_3])$ on $\tilde{E} \times_{\mathrm{Spec}(L)} F_0$. By functoriality (see corollary 3.90 and remark 3.72) we can transform a point on X to one on X''_* and evaluate it there instead of descending the Brauer class to X.

Working over the field L, which contains a 3rd root of unity, enables us to apply simpler formulas for computing local invariants. There is only a single lift of a $\mathbb{Q}_2$-point since the splitting degree of $L/\mathbb{Q}$ is 1 at 2.

The result is independent of that choice of a root $u = \sqrt{1 + 3t^4}$: we denote the completion of L at a (non-split) prime p by L_p – in particular this is the case for the non-split primes 2 and 3. Explicit calculations show that any solution of $x_0^4 + 3x_1^4 = 4x_2^4 + 9x_3^4$ in $\mathbb{Z}/2^5\mathbb{Z}$ has the property that $x_0^4 + 3x_1^4$ is a square in $(\mathbb{Z}/2^5\mathbb{Z})[\sqrt{-3}]$. We apply Hensel's lemma 2.89 to the polynomial $r := y^2 - (x_0^4 + 3x_1^4)$ in $\mathbb{Q}_2(\sqrt{-3})$. We have $r' = 2y$, and since $\mathbb{Q}_2(\sqrt{-3})/\mathbb{Q}_2$ is unramified, we get $k = \mathrm{val}(r') = 2$. Thus setting $m = 5 > 4 = 2k$ the conditions of Hensel's lemma are fulfilled, and we get a solution in $\mathbb{Z}_2[\sqrt{-3}]$, and since $\mathbb{Q}_2(\sqrt{-3}) \subset L_2$, also in L_2. This means that at a given 2-adic point the extension given by $u^2 = s^4 + 3t^4$ is split and we can take one of the isomorphic L_2-factors of $L_2[\sqrt{s^4 + 3t^4}] \cong L_{2,\sqrt{s^4+3t^4}} \oplus L_{2,-\sqrt{s^4+3t^4}}$ to work in.

Next we show that q_1 evaluated at a lift of a 2-adic point of X is a cube in L_2, and thus the local invariant of $(q_1, q_2)_3$ must be 0 over L_2. We use the corestriction map of remark 3.72 and the local invariant over $\mathbb{Q}_2$ is therefore also 0.

We apply Hensel's lemma to $y^3 - q_1(x)$ for 2-adic points x. Its derivative $3y^2$ has coefficients prime to 2, thus we only need to check modulo $2^1 = 2$. Now we let `MAGMA` calculate all solutions modulo 2, evaluate q_1 at these points and check, if it is a 3rd root in L_2. This turns out to be the case for any modulo 2 solution. Thus the local invariant at 2 is always 0.

For the prime 3 we similarly as for 2 prove that $x_0^4 + 3x_1^4$ is always a square in L_3 for $\mathbb{Q}_3$-points on X – it is even simpler because the valuation of the derivative is 0, so we may apply a smooth version of Hensel's lemma. Thus again we may work over L_3 rather than $L_3[\sqrt{x_0^4+3x_1^4}]$, and by the same reasons as for 2, the invariant does only change by a factor of 2 when computed over L_3.

For cyclic algebras of degree $n=3$ like $(q_1,q_2)_3$ we can compute the invariant by the local Hilbert symbol (= norm residue symbol) provided that the n-th root of unity is contained in the base field (see [**65**, Hasse invariant]). This condition is fulfilled, since $\zeta_{12}^4=\zeta_3\in L_3$. There are explicit formulas for the local Hilbert symbol (see, e.g., [**39**, VII.] or [**100**, V. (3.7) Theorem]).

We use the formula in [**100**, V. (3.7) Theorem], but with the formula for h as in [**39**, VII. (3.6)], which is called V there. There are some convergence issues involved in these calculations. The formulas involve computing Laurent series associated to $q_1(x),q_2(x)$ for x a 3-adic point and taking Laurent series residues of that series, followed by a $\bmod 3$-residue. This means that we have to compute the associated Laurent series only up to the precision where the $\bmod 3$-residue of the -1-power coefficient of the resulting series stabilizes. In [**134**] one can find bounds which apply in our case.

Then we need to test all 3-adic points. Since by Hensel's lemma any solution $\bmod 3^n$ for $n\geq 1$ lifts, we only need to consider solutions up to a finite precision. Since we want to test $3|x_1x_3$ and the coefficient of the terms which involve x_1,x_3 are $4x_1^4$ and $-9x_3^4$, hence have valuation below 2, we need to check solutions at least $\bmod 3^{(1+2)}$ to verify this congruence relation. Those variables whose terms in the defining equation $x_0^4+3x_1^4-4x_2^4-9x_3^4$ have coefficients with nontrivial valuation do not need to be determined up to full precision, e.g., for x_3 congruence $\bmod 3$ would be enough.

We need to take n big enough such that the evaluations of q_1,q_2 at any two different 3-adic points $a,b\in X(\mathbb{Q}_3)$ with $a\equiv b \mod 3^n$ give the same elements modulo $L_3^{(3)}$ when lifted. In order to do so we split $q_1=q_1'/q_1'',q_2=q_2'/q_2''$ into numerator and denominator and check whether $q_1'(a)/q_1'(b)\in L_3^{(3)}$, etc.

3 has valuation 8 in L_3, and η has valuation 1. Inspection of the coefficients of q_1,q_2 shows that the terms with coefficients of uniquely least valuation are exactly the u^3-terms for all 4 polynomials q_1',q_1'',q_2',q_2'' and the coefficients have all valuation 13. In other words if all u^3 have valuation 0 for all 3-adic points $a:=[x_0:x_1:x_2:x_3]$ up to precision n, which we call u^3-condition for the time being, then the valuations of $q_1'(a),q_1''(a),q_2'(a),q_2''(a)$ are all 13. Note that the equations were for the chart $x_0=x=1$ and otherwise we had to replace u by $ux^2=v$ in q_2. An easy 3-adic argument shows that any integral primitive 3-adic solution has an x_0 of valuation 0, so this does not interfere with the analysis.

If we determine 3-adic solution up to $\bmod 3^4$, then $u=\sqrt{x_0^4+3x_1^4},x=x_1,y=x_2,z=x_3$ are all determined $\bmod 3^2$. Thus simple comparison of valuations of coefficients shows that under u^3-condition $q_1'([x_0:x_1:x_2:x_3]),q_1''([x_0:x_1:x_2:x_3]),q_2'([x_0:x_1:x_2:x_3]),q_2''([x_0:x_1:x_2:x_3])$ are determined up to valuation $13+2\cdot 8$, thus the respective quotients are determined up to valuation $2\cdot 8$ and are of the form $d := 1 + 3^2 \cdot c$ for c in $\mathfrak{o}_{L_3}$. The proof of [**39**, I. Proposition (5.9)] tells us that such a d is in $L_3^{(3)}$.

To verify the u^3-condition it suffices to verify the analogous condition for u. To this end we compute all $\mathbb{Q}_3$-solutions $\mod 3^4$, taking into account that we do not need to compute x_1, x_3 up to full precision since their coefficient have nontrivial valuation, and then check the valuation of u. With `MAGMA` we can go through all the 22046/2 cases and see that the u^3-condition is fulfilled.

We still need to evaluate the invariant of $(q_1, q_2)_3$ at the 22046 solutions $\mod 3^4$. We do this with another `MAGMA`-script. There are 3-adic points not satisfying $3|x_1x_3$, but at all these points the invariant is either 1/3 or 2/3, and conversely at all other points the invariant is 0.

Since 3 is also non-split in $L/\mathbb{Q}$ the computed invariant is that of the corestricted class as remark 3.72 suggests, since the corestriction induces id on the invariants for a local field (see [**101**, (7.1.4) Corollary]).

At all primes except 3 the local invariant is 0. We used the cup product representation to easily bound the set of bad places for the cocycle. In general a result of Colliot-Théléne and Skorobogatov [**26**, Proposition 2.4. and Corollary 3.3.] using simply conectedness of K3 surfaces (see 2.44) shows that at all places except in our case 2, 3 the invariant is constant, but possibly non-zero. This would in our case also be enough to show the congruence relation for rational points.

The following table shows the correlation between the invariants over $\mathbb{Q}_3$ and the residue classes $r \equiv x_1x_3 \mod 3$ for the 22046 solutions $\mod 3^4$:

	inv = 0	inv = 1/3	inv = 2/3
$r = 0$	11960	0	0
$r = 1$	0	2508	2508
$r = 2$	0	2535	2535

Finally we conclude that all $\mathbb{Q}$-rational points p of the surface $X := \{x_0^4 + 3x_1^4 - 4x_2^4 - 9x_3^4 = 0\} \subset \mathbb{P}^3_{\mathbb{Q}}$ satisfy $\sum_{\mathfrak{p} \in \Omega_{\mathbb{Q}} \setminus \{(3)\}} \mathrm{ev}_{\mathfrak{p},[(q_1,q_2)_3]}(p) = 0$, thus the local invariant at 3 must be 0 too for any $\mathbb{Q}$-point p. We deduce by the above table that $3|x_1x_3$, and that this is completely explained by an obstruction to weak approximation by a transcendental 3-torsion element in $\mathrm{Br}(X)$.

Remark 4.2. In our case the invariants at each prime agree for all rational points. This is wrong in general as [**126**, proof of Proposition 7.1.] shows.
The transcendental 3-torsion obstruction in this example seems to dismiss only about 1/2 of the adelic points at least $\mod 3^4$ as the above table indicates. This observations might be interesting in the context of quantitative asymptotics with respect to some height function (see [**73**, VI.5.]), but we have no idea about how to interpret the data in this context.

4.2. Computing the Degree of an Isogeny

We are interested in the degree of the morphism $\mathrm{Jac}_{X'_*/A_*} \xrightarrow{p:=p_0\times p_1\times p_2} Y_0 \times_A Y_1 \times_A Y_2$. The degree is by definition the degree of the field extensions at the generic point. Base changing to the algebraic closure does not alter the degree, since the function fields of these varieties V satisfy $k(V) \cap \overline{\mathbb{Q}} = \mathbb{Q}$.

For morphisms of fibrations the degree of the whole morphism coincides with the degree of the induced morphism on the general fiber. We therefore may restrict to a fiber. Over an algebraically closed field, we also may change the coefficients to better suite our computations.

For the rest of this section $i \in \{0, 1, 2\}$ and the base field is $\mathbb{Q}^{alg}$ and we therefore omit the $\bar{\cdot}$. A fiber of X'_* is isomorphic to

$$C := \{x_0^4 + x_1^4 + x_2^4 = 0\} \subset \mathbb{P}^2.$$

We have the degree 2 cover to an elliptic curve

$$q_0 : C \xrightarrow{q_0} \{x_0^2 + x_1^4 + x_2^4 = 0\} =: F_0 \subset \mathbb{P}(2,1,1), [x_0 : x_1 : x_2] \mapsto [x_0^2 : x_1 : x_2]$$

ramified at the four points with $x_0 = 0$. We have analogous morphisms q_1, q_2.

Fix a point $P \in C$. According to [**23**, VII.2.] there is a unique embedding $C \hookrightarrow J$ into the Jacobian of C mapping P to the neutral element $O \in J$. We define the neutral elements of the F_i by $O_i := p_i(P)$. By the universal property of Jacobians [**23**, VII Proposition 6.1.], there are unique morphisms f_i of abelian varieties, i.e., additionally respecting the group law such that the diagram below commutes. They induce a morphism f to the product:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "q_i"'] \arrow[dr, phantom, "\circlearrowleft", near start] & J \arrow[dl, "f_i"] \\ F_i & \end{tikzcd} \qquad \begin{tikzcd} J \arrow[d, "f:=(f_0,f_1,f_2)"] \\ F_0 \times F_1 \times F_2 \end{tikzcd}$$

We want to prove that f is an isogeny (cf. [**58**, pp. 95/134]) of degree 8. The morphism f of abelian varieties is between varieties of equal dimension since the geometric genera of C respectively F_i are 3 respectively 1. Since the morphisms q_i are surjective, they induce surjective morphisms f_i on degree 0 divisors, and thus we only need to prove that a single fiber of a closed point in $F_0 \times F_1 \times F_2$ has multiplicity 8 in J. To this end we look at the following three commutative diagrams one for each $i \in \{0, 1, 2\}$, where S_3 is the group scheme associated to the symmetric group on 3 elements:

$$\begin{tikzcd}
(p_0, p_1, p_2) \arrow[rr, mapsto] \arrow[dddd, mapsto] & & \sum_j p_j \arrow[rr, mapsto] & & (f_k(\sum_j p_j))_k \arrow[dddd, mapsto] \\
& C^3 \arrow[r, two heads, "r"] \arrow[d, two heads, "q_i^3"'] & J \arrow[r, two heads, "f"] & F_0 \times F_1 \times F_2 \arrow[d, two heads, "\mathrm{pr}_i"] & \\
& F_i^3 \arrow[r, two heads, "\alpha_i"] \arrow[rr, two heads, bend right, "\Sigma_i"'] & F_i^3/S_3 \arrow[r, two heads, "\Sigma'_i"] & F_i & \\
& & & & \\
(q_i(p_j))_j \arrow[rr, mapsto] & & \langle q_i(p_j) \rangle_j \arrow[rr, mapsto] & & \sum_j q_i(p_j) = f_i(\sum_j p_j).
\end{tikzcd}$$

By general facts about quotients of powers by symmetric groups and about Jacobians (see [**23**, VII.3./5.]), we immediately see that α_i, r are finite morphisms of degrees 6 and that the q_i^3 are of degree 8. Σ_i is the sum map on the the triple power of the elliptic curves F_i^3, and Σ'_i is the induced map on the S_3-quotient. The diagram is clearly commutative.

We show that the preimage $f^{-1}(O_{F_0\times F_1\times F_2})$ has multiplicity 8. Using the above diagram we see that

$$r^{-1}(f^{-1}(O_{F_0\times F_1\times F_2})) = \bigcap_i (q_i^3)^{-1}(\alpha_i^{-1}(\Sigma_i'^{-1}(\mathrm{pr}_i(O_{F_0\times F_1\times F_2}))))$$

where we use scheme theoretic intersection, i.e., intersection respecting multiplicities. Since we know that r is of degree 6, it suffices to show that $r^{-1}(f^{-1}(O_{F_0\times F_1\times F_2}))$ has multiplicity $6 \cdot 8 = 48$ to prove that $f^{-1}(O_{F_0\times F_1\times F_2})$ has multiplicity 8. $f^{-1}(O_{F_0\times F_1\times F_2})$ might lay in the ramification locus of r, and $r^{-1}(f^{-1}(O_{F_0\times F_1\times F_2}))$ might not be 0-dimensional. So a priori there is no guarantee that this method succeeds, but the potential problems with ramification will be seen to not occur.

We apply intersection theory (see [**44**] or [**61**, Appendix A]) and therefore switch to cycle classes. All involved morphisms are flat and proper, thus pullback and pushforward are defined. To simplify notation, we implicitly apply the degree homomorphism from 0-dimensional cycles to $\mathbb{Z}$. By construction cycle classes respect intersection numbers. We thus have to compute the intersection number

$$r^*f^*[O_{F_0\times F_1\times F_2}] = \prod_i (q_i^3)^*\alpha_i^*\Sigma_i'^*[\mathrm{pr}_i(O_{F_0\times F_1\times F_2})] = \prod_i (q_i^3)^*\alpha_i^*\Sigma_i'^*[O_i].$$

We first compute the class $\Sigma_i'^*[O_i] = [\Sigma_i'^{-1}(O_i)]$. The computation is symmetric in the i, so we may assume $i = 0$ and drop the index i until we come back to the intersection product. First we show that $F^3/S_3 \xrightarrow{\Sigma'} F$ is a projective space bundle, i.e., it is a family of algebraically varying projective planes. We identify an elliptic curve with its Jacobian and a point p on it with the degree 0 divisor classes $p - O$. By $\langle p', p'', p''' \rangle$ we denote a symmetrized triple of points in F^3/S_3.

Let us proof that the fibers of Σ' are isomorphic to $\mathbb{P}^2$. Fix a $p \in F$ and look at the fiber $\Sigma'^{-1}(\{p\}) = \{\langle p', p'', p''' \rangle; p' + p'' + p''' = p\}$. Let $D := p + 2O$. For any point in the fiber of p we have $p'+p''+p'''-D=(f)$ for some rational function $f \in k(F)^*$. If we express this in linear systems, we have $k^*f \in |D| = \mathbb{P}(H^0_{zar}(F, \mathcal{L}(D)))$. We can reverse this, i.e., any f with $k^*f \in |D| = \mathbb{P}(H^0_{zar}(F, \mathcal{L}(D)))$ defines by requiring $p'+p''+p''' = (f)+D$ a point $\langle p', p'', p'''\rangle$ on the fiber $\Sigma'^{-1}(\{p\})$. So the fiber must be a projective space.

D is effective of degree 3, and a canonical divisor K is of degree 0, since F is of geometric genus 1 (see [**61**, IV. Example 1.3.3.]). Therefore $K-D$ is ineffective of degree -3. By Riemann-Roch for curves (see [**61**, IV. Theorem 1.3.]):

$$l(D) = l(D) - l(K-D) = \deg(D) + 1 - p_g = 3 + 1 - 1 = 3.$$

Thus $\dim(\Sigma'^{-1}(\{p\})) = \dim|D| = 3-1 = 2$, which establishes that the fibers are all projective planes.

Now we prove the relative version. For this look at the commutative diagram:

$$\begin{tikzcd}
F \times (F^3/S_3) \arrow[rr, "(\mathrm{id},\Sigma')"] & & F \times F \arrow[r, "\pi:(p,p')\mapsto p"] & F \\
F^3/S_3 \cong \{(p, \langle p', p'', p'''\rangle); p = p'+p''+p'''\} \arrow[u, hook] \arrow[rr, "(p,\langle p',p'',p'''\rangle)\mapsto p"] & & \Delta \arrow[u, hook, "p\mapsto(p,p)"'] \arrow[ur, "\mathrm{id}"] &
\end{tikzcd}$$

We need to replace the fixed point p by the diagonal Δ, O by $O' := F \times O$, and D by $D' := \Delta + 2O'$, and look for a rational function $f \in k(F \times F)^*$ with $k^*f \in \mathbb{P}(R^0\pi_*\mathcal{L}(D')) = \mathbb{P}(\pi_*\mathcal{L}(D'))$. As in [**61**, V. Proposition 2.2.] it suffices to show that $\pi_*\mathcal{L}(D')$ is a locally free sheaf to conclude that Σ' defines a projective space bundle. Let $p \in F$ be a fixed base point. Define for $j \in \mathbb{Z}$:

$$\phi^j := \phi^j(p) : R^j\pi_*\mathcal{L}(D') \otimes k(p) \to H^j_{zar}(F \times p, \mathcal{L}(D')\,|_{F\times p}) \cong H^j_{zar}(F, \mathcal{L}(D)).$$

Since ϕ^{-1} is the 0-map and $R^0\pi_* = \pi_*$, [**61**, III. Theorem 12.11.] yields the following implications:

(1) if ϕ^1 is surjective, it is an isomorphism,
(2) if ϕ^1 is surjective, then (ϕ^0 is surjective $\Leftrightarrow$ $R^1\pi_*\mathcal{L}(D')$ is locally free in a neighborhood of p),
(3) if ϕ^0 is surjective, it is an isomorphism,
(4) if ϕ^0 is surjective, then $\pi_*\mathcal{L}(D')$ is locally free in a neighborhood of p.

We have from the fiberwise analysis and Riemann-Roch that $h^1(\mathcal{L}(D)) = h^0(\mathcal{L}(K-D)) = 0$, and thus $H^1_{zar}(F, \mathcal{L}(D)) \cong 0$. This means that ϕ^1 must be surjective. By (1) ϕ^1 is an isomorphism, hence $R^1\pi_*\mathcal{L}(D') \otimes k(p) \cong 0$. Since $R^1\pi_*$ vanishes over all residue fields, hence on all stalks, we have global vanishing $R^1\pi_*\mathcal{L}(D') \cong 0$. In particular it is locally free. Thus we may apply (2) and get surjectivity of ϕ^0, hence by (3) it is an isomorphism. This means that after restriction to closed points we have by the above fiberwise argument that $\pi_*\mathcal{L}(D') \otimes k(p) \cong H^0_{zar}(F, \mathcal{L}(D)) \cong k(p)^3$. By (4) $\pi_*\mathcal{L}(D')$ is locally free and we have $\pi_*\mathcal{L}(D')(U) \cong k(U)^3$ for some neighborhood $U \ni p$. This is a local trivialization, and thus $F^3/S_3 \xrightarrow{\Sigma'} F$ is a projective space bundle.

Intersection numbers on smooth projective varieties are well defined on cycles up to rational equivalence. For smooth varieties such as F^3/S_3 the Chow group and the ring of cocycles are isomorphic as groups: $A_*(F^3/S_3) \cong A^*(F^3/S_3)$. Thus we may use the intersection product on cocycles as well as on cycles promoting the Chow group to the Chow ring. Additionally we tensor this last group by $\mathbb{Q}$ to be able to divide relations by integers and neglect the torsion subgroup which also does not affect intersection numbers. We denote the resulting group by $A(F^3/S_3)$.

Let $Z := \{\langle p', p'', p'''\rangle; O \in \{p', p'', p'''\}\} \subset F^3/S_3$ respectively its associated divisor. For the projective space bundle as above we have a relative twisting sheaf $\mathcal{O}(1)$ which corresponds to the sum of the divisor Z and a pullback of a divisor on F by Σ'. The relative twisting sheaf defines a cycle class $\tilde{c} := [\mathcal{O}(1)]$ in $A(F^3/S_3)$. For projective space bundles we have explicit formulas describing the Chow group (see [**61**, Appendix A.11.]), which yield

$$A(F^3/S_3) \cong \langle \tilde{c}^0, \tilde{c}^1, \tilde{c}^2 \rangle_{A(F)}$$

is free as an $A(F)$-module (even after $\otimes\mathbb{Q}$). Replacing $\tilde{c}$ by $c := [Z]$, which differs only by a pull back of a divisor of the base, amounts to a change of basis for the module $A(F^3/S_3)$ and we may compute in $A(F^3/S_3)$ via $A(F)$ as follows:

$$\forall [B] \in A(F^3/S_3) : [B] = 0 \Leftrightarrow \forall n \in \{0,1,2\} : \Sigma'_*[B] \cdot c^n = 0 \in A(F). \qquad (4.2.1)$$

The goal is to express $\sigma := \Sigma'^*[O] = [\Sigma'^{-1}O]$ by classes, which have geometrically a simpler description. Next to $c = [\{\langle p', p'', p'''\rangle; O \in \{p', p'', p'''\}\}]$ the other class involved is $T := [\{\langle p', p'', p'''\rangle; \#\{p', p'', p'''\} < 3\}]$. We compute for each of these three classes components according to (4.2.1).

c: We use the diagram

$$\begin{tikzcd} F^3 \arrow[r, "\alpha"'] \arrow[rr, bend left, "\Sigma"] & F^3/S_3 \arrow[r, "\Sigma'"'] & F, \end{tikzcd}$$

and do the following computation for $n \in \{0,1,2\}$ using $\deg(\alpha) = 6$:

$$\Sigma'_*((6c) \cdot c^n) = \Sigma'_*(6c^{n+1} \cdot [F^3/S_3]) = \Sigma'_*(c^{n+1} \cdot 6[F^3/S_3]) \overset{6[F^3/S_3] = \alpha_*[F^3]}{=}$$
$$\Sigma'_*(c^{n+1} \cdot \alpha_*[F^3]) \overset{\text{projection formula}}{=} \Sigma'_*\alpha_*((\alpha^*c)^{n+1} \cdot [F^3]) = \Sigma_*((\alpha^*c)^{n+1}).$$

Now denote by $\tilde{S}_0 := \{(O, p'', p'''); p'', p''' \in F\} \subset F^3$ a subvariety, and similarly for $\tilde{S}_1, \tilde{S}_2$. The associated classes are called S_0, S_1, S_2. We have $\alpha^{-1}(\{\langle p', p'', p'''\rangle; O \in \{p', p'', p'''\}\}) = \tilde{S}_0 \amalg \tilde{S}_1 \amalg \tilde{S}_2$, since the S_k are mapped $2:1$ on their image by α. We have $\alpha^*c = S_0 + S_1 + S_2$. We use Chow's moving lemma: we have $[O] = [\sum_{j=0}^m a_i p_i]$ where $a_j \in \mathbb{Z}$ and $p_j \in F \setminus \{O\}$ and get $S_0^2 = \sum_{j=1}^m a_j[\{(O, p'', p'''); p'', p''' \in$

$F\}][\{(p_j, p'', p'''); p'', p''' \in F\}] = \sum_{j=1}^{m}(a_i \cdot 0) = 0 = S_k^2.$

$$(\alpha^* c)^{n+1} = \begin{cases} S_0 + S_1 + S_2 & n = 0 \\ 2S_0S_1 + 2S_1S_2 + 2S_2S_0 & n = 1 \Rightarrow \\ 6S_0S_1S_2 & n = 2 \end{cases}$$

$$\Sigma_*(\alpha^* c)^{n+1} = \begin{cases} 0 & n = 0 \\ 6[F] & n = 1 \Rightarrow \Sigma'_* c^{n+1} = \\ 6[O] & n = 2 \end{cases} \begin{cases} 0 & n = 0 \\ [F] & n = 1 \\ [O] & n = 2. \end{cases}$$

The classes S_k are of dimension 2 and they are pushed forward into $A_2(F) \cong 0$, since $\dim(F) = 1$. Thus we get $\Sigma'_* c = 0$ for dimension reasons. In the other two cases we use the geometric description of the cycle classes, and intersect the representing subvarieties scheme theoretically, which gives the result stated in a straight forward manner.

σ: This case can be done by direct computations. The $n = 0$ term again is trivial for dimension reasons. For $n = 1$ we argue with the varieties underlying the cycle classes: $\Sigma'^{-1}O \cap \{\langle p', p'', p''' \rangle; O \in \{p', p'', p'''\}\}$ is completely contained in the fiber over O, and when pushed forward via Σ' this gives O, which is 0-dimensional. Now dimensions do not match, and according to the definition of pushforward on cycle classes (see [**61**, Appendix A.1.]) we get $\Sigma'_*(\sigma c) = 0$. For $n = 2$ we use the projection formula for projective space bundles and the result from the computation for c, and get $\Sigma'_*(c^2 \cdot \Sigma'^*[O]) = (\Sigma'_* c^2) \cdot [O] = [F] \cdot [O] = [O]$. In summary:

$$\Sigma'_*(\sigma c^n) = \begin{cases} 0 & n = 0 \\ 0 & n = 1 \\ [O] & n = 2. \end{cases}$$

T: We use the diagram

$$\begin{tikzcd} T_2 := F \times F \arrow[r, hook, "\Delta_{0,1} \times \mathrm{id}_2"'] \arrow[rr, bend left, "\alpha'"] & F^3 \arrow[r, "\alpha"'] & F^3/S_3 \arrow[r, "\Sigma'"'] & F \end{tikzcd}$$

As α' is generically $1:1$ onto its image, i.e., the degree is 1, we have $\alpha'_*([T_2]) = T$. By straightforward calculations on the level of subvarieties:

$$\alpha'^* c = (\Delta_{0,1} \times \mathrm{id}_2)^*(\alpha^*[\{\langle p', p'', p''' \rangle; O \in \{p', p'', p'''\}\}]) = \\ [\{(O, p); p \in F\}] + [\{(O, p); p \in F\}] + [\{(p, O); p \in F\}].$$

For dimension reasons $\Sigma'_*(Tc^0) = 0$. For $n = 1$ we have

$$\Sigma'_*(T \cdot c^1) = \Sigma'_*(\alpha'_*([T_2]) \cdot c) \overset{\text{projection formula}}{=} \Sigma'_* \alpha'_*([T_2] \cdot \alpha'^* c) = \\ \Sigma'_* \alpha'_*(2[\{(O, p); p \in F\}] + [\{(p, O); p \in F\}]) = \\ \Sigma'_*(2[\{\langle O, O, p \rangle; p \in F\}] + [\{\langle p, p, O \rangle; p \in F\}]) = 2[F] + 4[F] = 6[F].$$

Essentially Σ' induces the multiplication by 2 morphism on the subvariety $\{\langle p, p, O \rangle; p \in F\} \cong F$, thus the local degree is 4. For $n = 2$ we use Chow's moving lemma as above: $[\{(O,p);p\in F\}]^2 = 0 = [\{(p,O);p\in F\}]^2$.

$$\begin{aligned}\Sigma'_*(T\cdot c^2) = \Sigma'_*(\alpha'_*([T_2])\cdot c^2) &\overset{\text{projection formula}}{=} \Sigma'_*\alpha'_*([T_2]\cdot(\alpha'^*c)^2) = \\ &\Sigma'_*\alpha'_*(2(2[\{(O,p);p\in F\}][\{(p,O);p\in F\}])) = \\ &\Sigma'_*\alpha'_*(4[\{(O,O)\}])) = \Sigma'_*(4[\{\langle O,O,O\rangle\}]) = 4[O].\end{aligned}$$

In summary we have

$$\Sigma'_*(Tc^n) = \begin{cases} 0 & n=0 \\ 6[F] & n=1 \\ 4[O] & n=2. \end{cases}$$

Comparing the decompositions for c, σ and T yields the relation $\sigma = 3c - \frac{1}{2}T$.
Now we switch back to the indexed notation. We rewrite the intersection number:

$$r^*f^*[O_{F_0\times F_1\times F_2}] = \prod_i (q_i^3)^*\alpha_i^*\Sigma_i'^*[O_i] = \prod_i (q_i^3)^*\alpha_i^*(3c_i - \frac{1}{2}T_i).$$

We first compute $(q_i^3)^*\alpha_i^*c_i$ and $(q_i^3)^*\alpha_i^*T_i$ and then their intersection products. We set $\{\infty_1^i, \infty_2^i\} := q_i^{-1}(O_i)$ as preimage of the degree 2 cover q_i, where we without loss of generality assume that O_i is not in the ramification locus of any q_i. We denote by ι_i the obvious involution on C of which the F_i are the quotients.

To avoid unnecessary complicated notation we simply write, e.g., $[p,O,p]$ for $[\{(p,O_i,p)\};p\in F_i]$, or $[\infty_1^0,d,d']$ for $[\{(\infty_1^0,d,d')\} : d,d'\in C]$, and use, e.g., (p,O,p), or (∞_1^0,d,d') to denote representing subvarieties.

We have

$$\begin{gathered}(q_i^3)^{-1}\alpha_i^{-1}\{\langle p',p'',p'''\rangle; O_i\in\{p',p'',p'''\}\} = \\ (q_i^3)^{-1}((O,p,p')\amalg(p,O,p')\amalg(p,p',O)) = \\ (\infty_1^i,d,d')\amalg(\infty_2^i,d,d')\amalg(d,\infty_1^i,d')\amalg(d,\infty_2^i,d')\amalg(d,d',\infty_1^i)\amalg(d,d',\infty_2^i)).\end{gathered}$$

Since q_i^3 and α_i are flat morphisms we get for the classes by [**44**, Lemma 1.7.1]:

$$\begin{gathered}(q_i^3)^*\alpha_i^*c_i = c_i' := \\ [\infty_1^i,p,p'] + [\infty_2^i,p,p'] + [p,\infty_1^i,p'] + [p,\infty_2^i,p'] + [p,p',\infty_1^i] + [p,p',\infty_2^i].\end{gathered}$$

The preimage $\alpha_i^{-1}\{\langle p',p'',p'''\rangle; \#\{p',p'',p'''\} < 3\}$ is ramified of index 2 via α_i. Thus the scheme theoretic preimage has multiplicity 2 over the underlying reduced subvarieties. With the short hand notation $(\iota_0^{0/1}d,d,d')$ for $(d,d,d')\amalg(\iota_0(d),d,d')$ and its analogs we get

$$\begin{gathered}(q_i^3)^{-1}\alpha_i^{-1}\{\langle p',p'',p'''\rangle; \#\{p',p'',p'''\} < 3\} = \\ (q_i^3)^{-1}(2(p,p,p')\amalg 2(p,p',p)\amalg 2(p',p,p)) = \\ 2(\iota_i^{0/1}d,d,d')\amalg 2(\iota_i^{0/1}d,d',d)\amalg 2(d',\iota_i^{0/1}d,d),\end{gathered}$$

which consists of 6 components each of multiplicity 2. We get for the classes

$$(q_i^3)^*\alpha_i^*\frac{1}{2}T_i = [\iota_i^{0/1}d,d,d'] + [\iota_i^{0/1}d,d',d] + [d',\iota_i^{0/1}d,d] =: T_i'.$$

We now compute

$$r^*f^*[O_{F_0\times F_1\times F_2}] = \prod_i (q_i^3)^*\alpha_i^*(3c_i - \frac{1}{2}T_i) = \prod_i (3c_i' - T_i') = 3^3c'^3 - 3\cdot 3^2c'^2T' + 3\cdot 3c'T'^2 - T'^3,$$

where we used that by symmetry and abuse of notation

$$\begin{aligned} c'^3 &:= c_0'c_1'c_2', \\ c'^2T' &:= c_0'c_1'T_2' = c_0'T_1'c_2' = T_0'c_1'c_2', \\ c'T'^2 &:= T_0'T_1'c_2' = T_0'c_1'T_2' = c_0'T_1'T_2', \\ T'^3 &:= T_0'T_1'T_2'. \end{aligned}$$

We compute the intersection numbers for the first of the symmetric choices.

c'^3: We may compute this number by successively choosing 3 factors of the 6 summands of c' and determine cases which must give intersection 0. We choose from 6 summands for the first factor, and again from 6 for the second, but two of those choices must yield 0 by Chow's moving lemma and dimension reasons, since they are constant in the same component. For the third factor we may also choose from 6, but now $2\cdot 2$ yield trivial intersection by the same reason. Thus we have $6\cdot 4\cdot 2$ nontrivial combinations, which by symmetry yield the same intersection number. Clearly a nontrivial combination determines a unique point on C^3, so we have

$$c'^3 = 6\cdot 4\cdot 2\cdot 1 = 48.$$

c'^2T': We use the same method. Now we chose the first two factors from 6 choices, and the last one from 6 different choices. The first two factors give $6\cdot 4$ a priori nontrivial combinations. We may assume by symmetry that the choices fix the first two components, e.g. $[\infty_1^0, \infty_1^1, p]$. By sufficiently general choice of O we may assume that no ∞_l^0 is related to any $\infty_{l'}^1$ by powers of the involutions ι_2, so the intersection of $[\infty_1^0, \infty_1^1, p]$ with $[\iota_2^{0/1}d, d, d']$ vanishes for dimension reasons. This leaves 4 possibilities each fixing all three coordinates uniquely giving intersections of multiplicity 1.

$$c'^2T' = 6\cdot 4\cdot 4\cdot 1 = 96.$$

$c'T'^2$: We first choose two from the 6 summands of T_1', T_2' and then one of the 6 summands of c_3'. There are three cases to distinguish: a) the two T'-choices relate different pairs of components, b) they relate the same pairs of components but via a different relation, or c) they relate the same pairs by the same relation. For a) there are $6\cdot 4$ possibilities, for b) there are $3\cdot 3$ (note that the index i varies), and for c) there are $3\cdot 1$.
In case a) the choice of any of the 6 possibilities for c_3' fixed one component uniquely, and hence by the relation also the other two, thus we have intersection multiplicity 1.
In case b) the two relations force one component to be a fixed point of an involution. Each of these involutions has 4 fixed points (corresponding to the ramification points of the F_i as double covers of $\mathbb{P}^1$). The relation

then also fixes another component uniquely leaving one component to be fixed. This can be done with 2 from the 6 remaining choices the other give trivial intersection number.
In case c) similarly to case b) we need to choose from 2 of the c'-summands in order to get something nontrivial. It remains to determine the intersection number. By symmetry it suffices to compute $[d, d, d'][d, d, d'][d, d', \infty_1^2]$. Using projection to the first two components we see that the intersection number is equal to the selfintersection of the diagonal $C \cong \Delta \hookrightarrow C^2$ of a curve of geometric genus $p_g = 3$. Denote this inclusion by j. In the next calculation we refer to pages in **[61]** simply by "p. xxx":

$$\Delta \cdot \Delta \overset{\text{p. }358}{=} \deg_\Delta(j^*(\mathcal{L}(\Delta) \otimes_{\mathcal{O}_{C^2}} j_*\mathcal{O}_\Delta)) = -\deg_\Delta(j^*(\mathcal{L}(-\Delta) \otimes_{\mathcal{O}_{C^2}} j_*\mathcal{O}_\Delta)) \overset{\text{p. }115/145}{=}$$

$$-\deg_\Delta(j^*(\mathcal{J}(\Delta) \otimes_{\mathcal{O}_{C^2}} \mathcal{O}_{C^2}/\mathcal{J}_\Delta)) = -\deg_\Delta(j^*(\mathcal{J}(\Delta)/\mathcal{J}_\Delta^2)) \overset{\text{p. }175}{=} -\deg_\Delta \Omega_C$$

$$\overset{\omega_C = \bigwedge^{\dim(C)} \Omega_C \cong \Omega_C}{=} -\deg_\Delta \omega_C = -\deg_\Delta K_C \overset{\text{p. }296}{=} -(2p_g - 2) = 2 - 2 \cdot 3 = -4$$

In summary:

$$c'T'^2 = 6 \cdot 4 \cdot 6 \cdot 1 + 3 \cdot 3 \cdot 2 \cdot 4 + 3 \cdot 1 \cdot 2 \cdot (-4) = 192.$$

T'^3: We may choose three times from 6 choices. We distinguish three basic cases: a) all three choices relate the same coordinates, b) exactly two relate the same coordinates, and c) all relations occur at different pairs. Those cases occur $3 \cdot 2^3$ times for a), $18 \cdot 2^3$ times for b) and $6 \cdot 2^3$ times for c).
a) Since one coordinate is undetermined, the intersection locus has the wrong dimension, so it contributes 0.
b) The third relation relating a different pair determines the coordinate not involved in the other two relations uniquely once they are fixed. It remains to discuss the intersection behavior of the relations concerning the same pair. We make a distinction into two subcases α and β in analogy to b) and c) for $c'T'^2$. Subcase α occurs $18 \cdot 3 \cdot 2$ times, subcase β the remaining $18 \cdot 1 \cdot 2$ times. The analysis is analogous to that for $c'T'^2$ thus the intersection numbers are 4 for α and -4 for β.
c) Without loss of generality we may reduce combinatorial complexity by a factor of 6 and assume that the first relation is between the first two coordinate, the second between the last two, and the third relates the first and the last component. Let (p_0, p_1, p_2) be a point on C^3. Since the ι_i are involutions, we can rewrite the relations on the p_i as

$$p_1 = \iota_0^{a_0} p_0, p_2 = \iota_1^{a_1} p_1, p_0 = \iota_2^{a_2} p_2,$$

where $a_0, a_1, a_2 \in \{0, 1\}$. Since the ι_i act non-trivially on pairwise different coordinates of C they commute. So putting the three equations together we get

$$p_0 = \iota_2^{a_2} \iota_1^{a_1} \iota_0^{a_0} p_0.$$

We have $\iota_2\iota_1\iota_0 = \mathrm{id}$. In 2 of 8 cases we have $a_2 = a_1 = a_0 \in \{0, 1\}$ and get a trivial relation. In the remaining 6 cases we get a nontrivial fixed point condition, which can easily be seen to be fulfilled by 4 points for each combination. For the trivial relation case we give an exemplary calculation:

$$[d, \iota_0(d), d'][d', d, \iota_1(d)][\iota_2(d), d', d] = [d, \iota_0(d), \iota_1\iota_0(d)][\iota_2(d), d', d]$$
$$= [\iota_2(d), \iota_1(d), d][\iota_2(d), d', d] = [d', \iota_1(d), d][\iota_2(d), d', d][\iota_2(d), d', d]$$

Thus we have reduced this case to the situation of b) β and get intersection number -4.

In summary:

$$T'^3 = 0 + 18 \cdot 3 \cdot 2 \cdot 4 + 18 \cdot 1 \cdot 2 \cdot (-4) + 6 \cdot 2 \cdot (-4) + 6 \cdot 6 \cdot 4 = 384.$$

Combining all this we get

$$r^*f^*[O_{F_0 \times F_1 \times F_2}] = 3^3 c'^3 - 3 \cdot 3^2 c'^2 T' + 3 \cdot 3 c' T'^2 - T'^3 = 3^3 \cdot 48 - 3 \cdot 3^2 \cdot 96 + 3 \cdot 3 \cdot 192 - 384 = 48.$$

So indeed, since $\deg(r) = 6$, we have proved $\deg(f) = 8$.

4.3. Identifying Promising Cocycle Representatives

For this section the base field is always one of $\mathbb{Q}, \mathbb{Q}^{alg}$ or $\mathbb{C}$. Which one is used will be clear from the context. Over $\mathbb{C}$ we use standard terminology of Riemannian geometry like sheets of a covering.

Remember that we want to compute

$$H^1_{grp}(\widehat{\pi_1^{an}(A^{an})}, \mathrm{Pic}_{Y_i/A}[3]) \cong H^1_{\acute{e}t}(A, \underline{\mathrm{Pic}}_{Y_i/A}[3]),$$

where $\mathrm{Pic}_{Y_i/E}[3]$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$ as an abelian group. See corollary 3.51.

First we need to define the Galois action, i.e., the action of $\widehat{\pi_1^{an}(A^{an})}$ on $\mathrm{Pic}_{Y_i/A}[3]$. To this end we remember the comparison to singular cohomology in proposition 3.49

$$H^1_{grp}(\widehat{\pi_1^{an}(A^{an})}, \mathrm{Pic}_{Y_i/A}[3]) \cong H^1_{sing}(A^{an}, \widetilde{\mathrm{Pic}}_{Y_i/A}[3]),$$

where $\widetilde{\mathrm{Pic}}_{Y_i/A}[3]$ is the induced local system of finite abelian groups on A^{an}. As discussed in subsection 4.1.2 we may work with the analytic fundamental group $\pi_1^{an}(A^{an})$ itself. We determine the $\pi_1^{an}(A^{an})$-action on $\mathrm{Pic}_{Y_i/A}[3]$ by the permutation on the fibers induced by lifting a loop in A^{an} representing an element of $\pi_1^{an}(A^{an})$ to the cover given by the local system $\widetilde{\mathrm{Pic}}_{Y_i/A}[3]$. I.e., we determine the monodromy of $\widetilde{\mathrm{Pic}}_{Y_i/A}[3]$.

Effectively computing monodromy for algebraic curves is independently studied on its own, see, e.g., [**107**]. There are also some implementations, e.g., in `Maple`. None of them seemed to fit our needs, so we implemented a basic and heuristic version of monodromy computations suitable for our needs. The ideas are straightforward and well-known.

We outline the procedure in a general context giving remarks on our special situation. Let C be a smooth algebraic curve given as a ramified covering f to the complex projective line $\mathbb{P}^1$ by explicit algebraic equations missing a finite set of explicitly given points $P := \{p_1, \ldots, p_n\}$ of its completion. Let S be a finite local system given by algebraic equations in the coordinates on C.

First we need to determine generators for $\pi_1^{an}(C^{an})$. By Riemann-Hurwitz formula [**61**, IV. Corollary 2.4.] we can determine the geometric genus $p_g(C)$, if we know the ramification and the degree of f. These invariants are in general easy to determine from an explicit description of f. $\pi_1^{an}(C^{an})$ can be generated by $2p_g(C)+|P|$ generators as Riemannian geometry tells us.

We find a set of loops T representing those generators. We determine first a general enough base point $b \in C$ for our loops and project it to a base point $b' \in \mathbb{P}^1$. We project each $p \in P$ to $\mathbb{P}^1$, encircle the image by a small enough loop, connect that loop to b', and lift that loop to C. From the $\deg(f)$ many liftings select that loop that encircles b. The remaining $2p_g(C)$ generators are determined as follows: one determines the ramification points in $\mathbb{P}^1$, finds small enough loops around them, and connects it to b'. Each loop is lifted to several paths in C, and by standard facts on the fundamental group of a Riemann surface we can connect them to give loops on C, whose classes yield the missing generators. In general there are relations, but they are also well-known.

We discretize loops and paths on an affine chart of $\mathbb{P}^1$ simply by sequences of complex numbers. Similarly paths on C are represented by sequences of tuples of complex numbers corresponding to coordinates in an ambient projective space. The lifting of the loops in $\mathbb{P}^1$ according to the equations describing the cover is done in the same way as the determination of a lifting of loops on C according to the equations for S.

Let S_b be the fiber of the local system over b consisting of finitely many points. We lift the loops in T to paths in the local system S as described below. Let $\gamma \in T$ be one generator. If S is given by a degree d equation, then there are d lifts $\gamma_1, \ldots, \gamma_d$ to S each starting at another of the d points in S_b. Assume the paths γ_i to be maps from the unit interval. The map $S_b \to S_b, \gamma_i(0) \mapsto \gamma_i(1)$ is a permutation on S_b. This gives the permutation representation of $\pi_1^{an}(C^{an})$ that we are looking for.

We now describe the lifting. Given a path as a sequence of coordinates we put these coordinates into the defining equation of S and compute d solutions. If there are multiple solutions we get less than d distinct solutions and our implementation returns an error. However, a general path does not encounter these problems, so we may simply change the paths slightly to avoid such problems. Then we match successively the d solutions over the $i+1$st point in the path to the d solutions over the ith point. We do this by minimizing distances in the complex plane. Appending them accordingly, we get d new paths lifting the old one. Again we encounter potential numerical problems, e.g., if the step size is not small enough, then the distance minimizing heuristic might not give the correct path continuation. We have taken some precautions in the implementation about this, nevertheless it is only a heuristic and does not give provably correct output. See the remarks in the code section B.2.

Concretely in our situation $A = C$ and $S = \widetilde{\mathrm{Pic}}_{Y_i/A}[3]$. We study $S' := \widetilde{\mathrm{Pic}}_{Y_i/A}[3] \setminus \{O\}$ instead of S, since coordinates of S' are given in a simpler form, namely by the 3rd division polynomial for the x-coordinate of the relative elliptic curve in Weierstrass normal form and the y-coordinate by the Weierstrass equation applied to these x-coordinates. All the monodromy data is encoded in S' and extracted to give the monodromy on S as explained in subsection 4.1.2.

Combining the group actions for the Y_i, we get the action for X'. We compute the cohomology group $H^1_{grp}(\pi_1^{an}(A^{an}), \mathrm{Pic}_{X'/A}[3])$ using standard commands in `MAGMA`. The representing cocycles are in general quite big suggesting large étale covers as input data to compute a cocycle representation of potentially nontrivial Brauer group elements. This is not sufficient for practical purposes.

We need a further lemma from group theory.

LEMMA 4.3. *Let G be a finitely generated group and N a subgroup of finite index. Then N is also finitely generated. Moreover if G is given by an explicit finite set of generators and N is given by an explicit finite coset table, then we can effectively compute a finite set of generators for N.*

PROOF. See, e.g., [**110**, Lemma 7.56. and Exercise 11.47.]. The proof there is effective albeit this is not stated. $\square$

From now on we denote by $G := \pi_1^{an}(A^{an})$ the finitely generated group, $M := \mathrm{Pic}_{X'/A}[3]$, and the action of $g \in G$ on $m \in M$ by $g.m$, an action from the right. Since G is a finitely generated group and M is finite, the action has to factor through a finite quotient Q_{triv}, because there exists a matrix representation (in general a permutation representation) $G \xrightarrow{\rho} \mathrm{Gl}(M)$ into a finite group, so we can apply lemma 4.3. In order to find that quotient we only need to find the kernel N_{triv} of ρ and take the quotient Q_{triv} by that kernel:

$$0 \to N_{triv} = \ker(\rho) \xrightarrow{\pi} G \to Q_{triv} \to 0.$$

This is all implemented in `MAGMA`.

Now we give a procedure to find a relatively small quotient Q such that a given $[f] \in H^1_{grp}(G, M)$ is in the image of the associated injective map $H^1_{grp}(Q, (\pi^* M)^N) \xrightarrow{\inf} H^1_{grp}(G, M)$. For this let S be a finite set of generators of G and $f \in \mathrm{Map}(G, M)$ a twisted homomorphism (= crossed homomorphism, see [**101**, p. 16]) representing $[f]$. Let N_{triv}, Q_{triv} be defined as above.

We first construct a subgroup $N \subset N_{triv}$. As above by lemma 4.3, N_{triv} is a finitely generated subgroup, and we may assume that we have effectively found a finite set of generators T_{triv}. When restricting f to N_{triv} we get a group homomorphism $f' \in \mathrm{Map}(N_{triv}, M)$. Since the twisting group action is trivialized, N_{triv} is finitely generated, and M is still finite, we see by lemma 4.3 that $N' := \ker(f')$ is effectively finitely generated by a set of generators T. As a kernel N' is a normal subgroup of N_{triv}, which in turn is a normal subgroup of G, but being a normal subgroup is not transitive. Let R be a coset table for N, efficitively computable by the

Todd-Coxeter algorithm (see [**110**, Theorem 11.8.]), since N has finite index. The normalization $N := \bigcap_{r\in R} rN'r^{-1} \subset N'$ is a normal subgroup of G of finite index with finite set of generators.

Now set $Q := G/N$ again effectively computable by Todd-Coxeter algorithm, and denote its elements by $[g]$ for $g \in G$. We define $\tilde{f} \in \mathrm{Map}(Q, M)$ by $\tilde{f}([g]) := f(g)$. Let $r \in N \subset N_{triv}$ be arbitrary. By construction of N and $\tilde{f}$ we have $f(r) = 0 = \tilde{f}([r])$. In general for an arbitrary $g \in G$ it holds $f(rg) = r.f(g) + f(r) = f(g) + 0 = f(g)$ and $f(gr) = g.f(r) + f(g) = g.0 + f(g) = f(g)$. Thus $\tilde{f}$ is well defined and clearly a twisted homomorphism.

Proposition 4.4. *Let G be a group with finite set of generators S acting on a finite abelian group M, e.g., by permutation action, and $f \in \mathrm{Map}(G, M)$ a twisted homomorphism. Then there is an effectively constructible finite quotient Q of G such that $[f] \in \mathrm{im}(H^1_{grp}(Q, (\pi^*M)^N) \xrightarrow{\inf} H^1_{grp}(G, M))$ and an effectively constructible twisted homomorphism $\tilde{f} \in \mathrm{Map}(Q, (\pi^*M)^N)$ representing the preimage.*

Proof. This is obvious in light of the argument before the statement of the proposition. □

Since Q was tailored for f, it is reasonable to assume that it is a comparably small quotient, that exhibits $[f]$ via $\tilde{f}$ as a nontrivial cohomology class. The constructed Q is probably in most cases still not optimal, since the constructed N is inside N_{triv}, thus $(\pi^*M)^N = \pi^*M$, and it is plausible that there are some cocycles not having all of M as its image. Thus we may want to enumerate the finitely many normal subgroups of Q, apply the Lyndon-Hochschild-Serre spectral sequence 3.60, and test whether $[f]$ can be represented by a cohomology class for a smaller quotient.

In fact in the implementation we follow this alternative approach. First we generate a finite quotient Q_{tot} of G which is big enough to represent the set of generators for the cohomology group computed with a `MAGMA`-internal command. During this we identify generators that have already small representations by a Q_i. Then we search for smaller quotients Q_{small} that have nontrivial cohomology. We identify such a Q_{small} to be isomorphic to S_3.

In general another approach might be better: even if we have a set of generators for the whole cohomology group, the generators itself might not be representable in a small quotient. But there might be nontrivial combinations of the generators of the cohomology group that are representable in a small quotient, and if we systematically screen small quotients Q of Q_{tot}, we can identify cohomology classes representable by small quotients, which we would not detect, if we only would work with the generators itself. Since in our situation we get a promising $Q_{small} \cong S_3$, we do not need to apply this more general method.

We close this section with some remarks on the transition from an explicit cocycle in $H^1_{grp}(Q, (\pi^*M)^N)$ to an étale cocycle in $H^1_{ét}(A, \underline{\mathrm{Pic}}_{X'/A}[3])$. When discussing the example, we already mentioned the general method using moduli of curves and cutting out moduli of Q-covers for the given base curve using effective invariant theory (see [**2**] and [**35**]). We discussed our approach in the special situation of the example, when searching for degree 3 cyclic extensions. This approach can be generalized to identify arbitrary degree n cyclic algebras as outlined in [**80**, Lemma 2.]. Another method using class field theory for general abelian extensions is given in [**23**, VII.9.]. The focus there is on unramified extensions of the base curve in accordance with our needs to produce unramified Brauer group elements. Ramified covers are also mentioned in [**23**, VII. Proposition 9.7.]. Koch in [**102**, pp. 158/190] gives some remarks on constructing Riemannian surfaces covering another Riemannian surface with prescribed Galois group. Algebraically this corresponds by GAGA to unramified covers of curves with given Galois group. This problem can be considered the geometric analog of an arithmetic problem related to the embedding problem. See [**101**, IX.4.] for details.

The approach of Kresch and Tschinkel in [**80**] for cyclic covers classifies the potential coverings by an abelian group. In order to find the covering which gives rise to the étale cover involved in representing a group cohomology cocycle as an étale cocycle, it is advisable to explicitly construct only étale covers for generators of this abelian groups. We may similarly as above use numerical path following algorithms to compute the monodromy action for these coverings for the generators of $\pi_1^{an}(A^{an})$ after the base change E/A) and compare it to the monodromy action encoded in the group cohomology cocycle for Q. Since the coverings are cyclic this is essentially linear algebra for $\mathbb{Z}/n\mathbb{Z}$-modules. We may determine the linear combination of generators of the abelian group of cyclic étale covers which gives rise to the cover involved in the étale cocycle corresponding to the given $[f] \in H^1_{grp}(Q, (\pi^* M)^N)$.

This approach, valid for cyclic groups, can clearly be generalized to solvable groups, when we split up the abelian factors for a composition series in cyclic factors and apply the above strategy successively. This method was also used in reducing the S_3-cover in our example to an $\mathbb{Z}/2\mathbb{Z}$-cover $u = \sqrt{1 + 3t^4}$ and a cyclic $\mathbb{Z}/3\mathbb{Z}$-cover.

4.4. An Attempt Using an Alternative Fibration

As mentioned in section 4.1, we first tried to compute transcendental 3-torsion elements in $\mathrm{Br}(X)$ using the elliptic fibration given in [**69**]. We give a sketch why this approach is not viable using the tools from above. At first we are only interested in the geometric situation, i.e., coefficients do not matter. We make some remarks about the field of definition of 3-torsion in the Picard group for the diagonal quartic given by $x_0^4 - x_1^4 - x_2^4 + x_3^4$. We use the terminology of [**69**] for easier comparison.

The fibration is given in proposition 2.114(2). After simple manipulations we arrive at the fibration given by

$$\mathbb{A}^1 \setminus \{5pt\} \times \mathbb{P}(2,1,1) \supset C := \{v^2 = (u^2 - t^3 s^2)(tu^2 - s^2)\} \to \mathbb{A}^1 \setminus \{5pt\}.$$

This fibration has no section, but its Jacobian, which is a relative elliptic curve, clearly has one. The Jacobian of C can be identified with $E := \{y^2 - x(x+1)(x+c^2) = 0\}$ where $c = (t^2-1)/(t^2+1)$ via the degree 2 covering τ defined by $x = (t^2-1)^2u^2/v^2, y = (t(t^2-1)^2u(u^4-t^2))/((t^2+1)v^3)$. Simply adjoining $\sqrt{t}$ enables an isomorphism of the base change of C with its Jacobian:

$$\begin{tikzcd}
C \arrow[d] \arrow[dd, bend right=60, "\tau"'] & C' \arrow[l, "\sqrt{t}"'] \arrow[d] \arrow[dd, bend left=60, "\cong"] \\
\mathbb{A}^1 \setminus \{5pt\} \arrow[d, bend right] & \mathbb{A}^1 \setminus \{9pt\} \arrow[l, "\sqrt{t}"'] \arrow[u, bend left] \arrow[d, bend right] \\
E \arrow[u] & E' \arrow[l, "\sqrt{t}"'] \arrow[u]
\end{tikzcd}$$

The square formed by the right most isomorphism, τ and the adjunctions of $\sqrt{t}$ does not commute. Everything else does.

To apply the same method as above we need to get a description of the 3-torsion of the fibration $E \to \mathbb{A}^1 \setminus \{5pt\}$. Instead of directly computing the necessary base curve extension directly, we simplify this fibration first. Then we compute the base extension B over which 3-torsion is defined and take the fiber product with the original Jacobian fibration. The reduction is done by $\mathbb{A}^1 \setminus \{5pt\} \to \mathbb{P}^1 \setminus \{3pt\}$ given by $t = c^2 = C^4$. This is a degree 4 morphism. All squares in this diagram are fibered products:

$$\begin{tikzcd}
 & \mathbb{P}^1 \setminus \{3pt\} & & \\
\mathbb{A}^1 \setminus \{5pt\} \arrow[ur, "deg=4"] & & & B \arrow[ull, "\chi"'] \\
 & E'' \arrow[uu] & \tilde{B} \arrow[ull] \arrow[ur] & \\
E \arrow[uu] \arrow[ur] & & & E''' \arrow[ull] \arrow[uu] \\
 & & \tilde{E} \arrow[ull] \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

We can look at the generic point for the above base curves and compute the degree of the extension $k(B)/k(\mathbb{P}^1 \setminus \{3pt\})$ using `MAGMA`. Over $\mathbb{Q}$ the degree is 48 and over $\mathbb{Q}(\zeta_3)$ it is 24, where ζ_3 is 3rd root of unity. Unfortunately this is the only number field extension contained in $k(B)/k(\mathbb{P}^1 \setminus \{3pt\})$. We know the function field extension explicitly and thus may compute B using `MAGMA` by iteratively adjoining defining equations, build an affine model, projectivize it in a trivial way, and blow up respectively normalize the resulting singularities of the compactification.

One can also try to compute the ramification divisor R for χ and use the Riemann-Hurwitz formula [**61**, IV. Corollary 2.4.]. We get $\deg(R) = 52$ and that the geometric genus of B is $p_g = 3$, which agrees with the `MAGMA`-computations, which also gives explicit equations for B as a plane quartic. The Hurwitz formula for the degree 4 cover $\tilde{B} \to B$ implies that the geometric genus is $\tilde{p}_g = 25$. In any case $\tilde{B}$ has to big a genus to compute with its Jacobian. Hence the fibers are still elliptic curves and easy to compute in, but to apply the method from above we also needed the base curve to have an accessible Jacobian.

In section 4.1 we had fibers of geometric genus 3, but the base was of geometric genus 1, and most of the field/curve extension to define all 3-torsion was within the degree 16 algebraic extension $L/\mathbb{Q}$. This makes the fibration which we finally used accessible for computations.

4.5. Remarks on Transcendental 2-Torsion

The case of transcendental 2-torsion was completely settled by Ieronymou in [**69**]. We want to outline an alternative approach.

According to Demazure in [**33**, V.4.] there is a correspondence between del Pezzo surfaces of degree 2 and smooth plane quartic curves. Twice the anticanonical divisor of a del Pezzo surfaces of degree 2 gives a double cover onto the projective plane ramified in a quartic. Under this correspondence the 56 lines (that is rational curves) correspond to the 28 bitangents of the associated plane quartic.

Using the fibration of section 4.1 in genus 3 curves isomorphic to plane quartics, we would need to compute 2-torsion in the Jacobian. This is again too hard a task in general. Here we may use a different trick. Denote by B a plane quartic. Clearly the difference of two hyperplane sections is a rationally trivial divisor, because it is the principle divisor associated to the quotient of the two linear forms defining the hyperplane sections. If we take the two hyperplane sections to be bitangents to a quartic, we get with multiplicities $(H \cap B) = 2P + 2Q, (H' \cap B) = 2P' + 2Q'$. Further if we take the difference of the halfs, then we get a 2-torsion divisor $(H \cap B)/2 - (H' \cap B)/2 = P + Q - P' - Q'$. Now each bitangent is double covered by two rational curves on the del Pezzo surface associated to B. If $C, \tilde{C}$ are these two curves over H, then we have $(C \cap B) = (\tilde{C} \cap B) = P + Q = (H \cap B)/2$, and analogously for $P' + Q'$. It turns out that the $(C \cap B) - (C' \cap B)$ actually generate the whole 2-torsion subgroup of the Jacobian of B.

Now the rational curves on degree 2 del Pezzo surface are fairly well understood. Using this we may replace the 2-torsion in a Jacobian of relative dimension 3 by the rational curves in a family of degree 2 del Pezzo surfaces. Over $\mathbb{C}$ we can apply path following techniques to determine a permutation representation of the rational curves in terms of loops generating the fundamental group of the pointed base space. From this we get a representation of $H^1_{ét}(A, \underline{\mathrm{Pic}}_{X'/A}[2])$. We could go on as above in computing sheaves of Azumaya algebras representing nontrivial transcendental elements of $\mathrm{Br}(X)[2]$.

APPENDIX A

Effective Lifting of 2-Cocycles for Galois Cohomology

A.1. Motivation and introduction

The vanishing of certain Galois cohomology groups of function fields and number fields, which are of interest in arithmetic geometry, is quite well established: e.g., there is Hilbert's Theorem 90 which gives the effective vanishing of $H^1(K/k, K^*)$ for any finite Galois extension, or Tsen's Theorem which when applicable implies the analogous vanishing of H^2. The goal is to provide effective and detailed versions of these results. We will describe the algorithms that will reduce the problem of exhibiting a cocycle explicitly as a coboundary if possible to solving norm equations. We are concerned specifically with the following problem:

Problem A.1. Given a finite group G and a G-module A, determine for an arbitrary cohomology class $[f] \in H^2(G, A)$ represented by an explicit 2-cocycle f of the standard cochain complex whether $[f] = 0$ and if so find a 1-cochain f' such that $\partial^1 f' = f$.

The analogous problem for r-cocycles (with $r \in \mathbb{N}$) will be denoted by (A.1_r). We prove

Theorem A.2. *Let G be the Galois group of a Galois extension K/k and A the usual G-module $K^* = K \setminus \{0\}$ both effectively given. Then we can effectively reduce Problem (A.1) for G and A to solving norm equations $N_{\tilde{K}/\tilde{k}}(x) = a$ for cyclic intermediate field extensions $\tilde{K}/\tilde{k}$ and $a \in \tilde{k}^*$.*

There will be three sections devoted to reduction steps introducing well-known results and general tools. The first of them relates Problem (A.1) in the case G is cyclic to solving norm equations. The second reduces Problem (A.1) for a general G to the case of finite p-groups, for various prime numbers p. The last of them reduces Problem (A.1) for G solvable to cyclic groups (and is presented last since it uses ingredients of the other reduction steps). This section is the only one depending strongly on the special case of Galois cohomology, specifically Hilbert's Theorem 90. A final section summarizes the results in a proof of Theorem (A.2). Additionally we show how to compute invariants in local fields explicitly, focusing on an application for number fields:

Proposition A.3. *Let K/k be a finite Galois extension of number fields with Galois group Γ. Let v be a place of k and $f \in \mathrm{Map}(\Gamma^2, K^*)$ an explicitly given 2-cocycle. We can compute $\mathrm{inv}_v[f] \in \mathbb{Q}/\mathbb{Z}$ effectively.*

We give an overview on the context of our results. In arithmetic geometry computing local invariants as in Proposition A.3 is used in Brauer-Manin obstruction. An instance of this setting in [**145**]. In [**146**] one finds an effective but rather indirect approach using Hensel lifting on Brauer-Severi varieties. For the special case of Hilbert symbols there are explicit formulas based on convergent power series (cf. [**148**, V. (3.7) Theorem] or [**140**, VII]). Effective bounds for the rate of convergence in the case of Hilbert symbols of order p over $\mathbb{Q}_p$ can be found in [**152**] as conferred by a referee. Our Proposition A.3 uses solubility of norm equations over local fields and is effective in full generality. Ramification in the associated Galois extensions increases only the precision needed in solutions of these norm equations and bounds for this can be read off from [**140**, III.1].

Another application is in minimizing representations as another referee pointed out. Recently Fieker in [**142**] gave an algorithm to solve this problem for number fields by reducing to Problem (A.1) and then solve S-unit equations. We give a comparison with our method in Remark A.14.

The results of the following three sections are well known at least on the level of cohomology ([**137**], [**151**] for some references) and partially also on the level of representing complexes. We provide all of the rather involved formulas needed for algorithms on the level of complexes. In particular the formulas referred to in Corollary A.5, Proposition A.8 and Proposition A.10 have not been found by us in the literature. It is also well known that formulas such as presented here can be applied as they are in the last part. The authors contribution is to actually do so.

A.2. Some complexes: from cyclic groups to norm equations

Let G be a finite group and A a G-module both effectively given and denote the action of $g \in G$ on $a \in A$ by $g.a$.

Recall that the "standard complex" in homogeneous form resolves the trivial G-module $\mathbb{Z}$ by the free G-modules $\mathbb{Z}[G^{r+1}]$ endowed with diagonal action given by $g.(g_0, \ldots, g_r) = (gg_0, \ldots, gg_r)$ (cf. [**137**, I.5]). Here and later on we omit the augmentation map to $\mathbb{Z}$:

$$\begin{aligned} SR := \quad & \mathbb{Z}[G^1] \xleftarrow{\partial_0} \mathbb{Z}[G^2] \xleftarrow{\partial_1} \mathbb{Z}[G^3] \xleftarrow{\partial_2} \cdots, \\ & \partial_r : \mathbb{Z}[G^{r+2}] \to \mathbb{Z}[G^{r+1}], \quad \sum_{\gamma=(g_0,\ldots,g_{r+1})\in G^{r+2}} c_\gamma \gamma \mapsto \\ & \sum_{k=0}^{r+1} (-1)^k \sum_{\gamma=(g_0,\ldots,g_{r+1})\in G^{r+2}} c_\gamma (g_0, \ldots, g_{k-1}, g_{k+1}, \ldots, g_{r+1}). \end{aligned}$$

When $G = \langle \alpha \rangle$ is cyclic of order $n \in \mathbb{N}$ we will also use the "efficient complex" ([**137**, I. (6.3)]), where the x_r serve as symbols to reference the position in the complex:

$$ER := \quad \mathbb{Z}[G]x_0 \xleftarrow{d_0} \mathbb{Z}[G]x_1 \xleftarrow{d_1} \mathbb{Z}[G]x_2 \xleftarrow{d_2} \cdots,$$

$$d_r((\sum_{g\in G} c_g g)x_{r+1}) := \begin{cases} ((\sum_{g\in G} c_g g) - (\sum_{g\in G} c_g \alpha g))x_r & r \text{ even},\\ (\sum_{i=0}^{n-1}\sum_{g\in G} c_g \alpha^i g)x_r & r \text{ odd}.\end{cases}$$

We give a quasi-isomorphism of these two complexes by specifying morphisms of chain complexes $\tau = (\tau_r)_{r=0}^{\infty} : ER \to SR$ and $\sigma = (\sigma_r)_{r=0}^{\infty} : SR \to ER$ and a chain homotopy $h = (h_r)_{r=1}^{\infty}$, satisfying $\sigma \circ \tau = \mathrm{id}$ and $\mathrm{id} - \tau_r \circ \sigma_r = h_r \circ \partial_{r-1} + \partial_r \circ h_{r+1}$. This diagram illustrates the situation:

$$\begin{tikzcd}
\mathbb{Z}[G^1] \arrow[d, "\sigma_0"] \arrow[ddr, "h_1"] & \mathbb{Z}[G^2] \arrow[l, "\partial_0"'] \arrow[d, "\sigma_1"] \arrow[ddr, "h_2"] & \mathbb{Z}[G^3] \arrow[l, "\partial_1"'] \arrow[d, "\sigma_2"] \arrow[ddr, "h_3"] & \mathbb{Z}[G^4] \arrow[l, "\partial_2"'] \arrow[d, "\sigma_3"] & \cdots \arrow[l] \\
\mathbb{Z}[G]x_0 \arrow[d, "\tau_0"] & \mathbb{Z}[G]x_1 \arrow[l, "d_0"'] \arrow[d, "\tau_1"] & \mathbb{Z}[G]x_2 \arrow[l, "d_1"'] \arrow[d, "\tau_2"] & \mathbb{Z}[G]x_3 \arrow[l, "d_2"'] \arrow[d, "\tau_3"] & \cdots \arrow[l] \\
\mathbb{Z}[G^1] & \mathbb{Z}[G^2] \arrow[l, "\partial_0"'] & \mathbb{Z}[G^3] \arrow[l, "\partial_1"'] & \mathbb{Z}[G^4] \arrow[l, "\partial_2"'] & \cdots \arrow[l]
\end{tikzcd} \tag{A.2.1}$$

Proposition A.4. *For a finite cyclic group $G = \langle\alpha\rangle$ of order n we get a quasi-isomorphism as in (A.2.1):*

$$\sigma_{2m}(\alpha^{\sum_{k=1}^{0} i_k}, \ldots, \alpha^{\sum_{k=1}^{2m} i_k}) := \begin{cases} (-1)^m e x_{2m} & \textit{if } \bigwedge_{k=1}^{m}(i_{2k-1} + i_{2k} \geq n),\\ 0 & \textit{otherwise;}\end{cases}$$

$$\sigma_{2m+1}(\alpha^{\sum_{k=1}^{0} i_k}, \ldots, \alpha^{\sum_{k=1}^{2m+1} i_k}) := \begin{cases} (-1)^{m+1}(\sum_{j=0}^{i_1-1}\alpha^j)x_{2m+1} & \textit{if } \bigwedge_{k=1}^{m}(i_{2k} + i_{2k+1} \geq n),\\ 0 & \textit{otherwise;}\end{cases}$$

$$\tau_{2m}(e x_{2m}) := (-1)^m \sum_{(i_k)_{k=1}^m = (0)_{k=1}^m}^{(n-1)_{k=1}^m} (e, \alpha^{\sum_{k=1}^{1} i_k}, \alpha^{1+\sum_{k=1}^{1} i_k}, \ldots, \alpha^{\sum_{k=1}^{m} i_k}, \alpha^{1+\sum_{k=1}^{m} i_k});$$

$$\tau_{2m+1}(e x_{2m+1}) := (-1)^{m+1} \sum_{(i_k)_{k=1}^m = (0)_{k=1}^m}^{(n-1)_{k=1}^m} (\alpha^{\sum_{k=1}^{0} i_k}, \alpha^{1+\sum_{k=1}^{0} i_k}, \ldots, \alpha^{\sum_{k=1}^{m} i_k}, \alpha^{1+\sum_{k=1}^{m} i_k});$$

$$h_1(e) := -(e,e);\ \ h_2(e, \alpha^{i_1}) := (e, \alpha^{i_1}, \alpha^{i_1}) - \sum_{j=0}^{i_1-1}(e, \alpha^j, \alpha^{j+1});$$

$$\textit{for } k \geq 3: h_k(e, \alpha^{i_1}, \alpha^{i_1+i_2}, \ldots, \alpha^{i_1+\cdots+i_{k-1}}) := (-1)^k\big((e, \alpha^{i_1}, \ldots, \alpha^{i_1+\cdots+i_{k-1}}, \alpha^{i_1+\cdots+i_{k-1}})$$
$$- \sum_{j=i_1+\cdots+i_{k-2}}^{i_1+\cdots+i_{k-1}-1}(e, \alpha^{i_1}, \ldots, \alpha^{i_1+\cdots+i_{k-2}}, \alpha^j, \alpha^{j+1})\big)$$
$$+ \begin{cases} \sum_{j=0}^{n-1}(h_{k-2}(e, \ldots, \alpha^{i_1+\cdots+i_{k-3}}), \alpha^j, \alpha^{j+1}) & \textit{if } i_{k-2} + i_{k-1} \geq n,\\ 0 & \textit{otherwise.}\end{cases}$$

PROOF. Let us prove that $(\sigma_r)_{r=0}^{\infty}$ constitutes a chain map. For this we prove for $m \in \mathbb{N}_0$

$$d_{2m}\sigma_{2m+1} = \sigma_{2m}\partial_{2m} \qquad \text{provided that} \tag{A.2.2}$$

$$d_{2m-1}\sigma_{2m} = \sigma_{2m-1}\partial_{2m-1}. \tag{A.2.3}$$

We call this the even case. It suffices to show equality on elements of the form $c := (e, \alpha^{i_1}, \alpha^{i_1+i_2}, \dots, \alpha^{\sum_{j=1}^{2m+1} i_j}) \in \mathbb{Z}[G^{2m+2}]$ for $i_1, i_2, \dots, i_{2m+1} \in \{0, 1, \dots, n-1\}$. We have:

$$c' := d_{2m}\sigma_{2m+1}(c) = d_{2m} \begin{cases} (-1)^{m+1}(\sum_{j=0}^{i_1-1} \alpha^j)x_{2m+1} & \text{if } \bigwedge_{k=1}^{m}(i_{2k} + i_{2k+1} \geq n), \\ 0 & \text{otherwise} \end{cases} =$$

$$\begin{cases} (-1)^{m+1}(e - \alpha^{i_1})x_{2m} & \text{if } \bigwedge_{k=1}^{m}(i_{2k} + i_{2k+1} \geq n), \\ 0 & \text{otherwise.} \end{cases}$$

For a suitable $\lambda_{(i_j)_{j=1}^{2m+1}} \in \mathbb{Z}$ we get:

$$c'' := \sigma_{2m}\partial_{2m}(c) = \sigma_{2m}(\sum_{k=1}^{2m+2} (-1)^{k+1}((\alpha^{\sum_{j=1}^{l-1} i_j})_{l=1}^{k-1}, (\alpha^{\sum_{j=1}^{l-1} i_j})_{l=k+1}^{2m+2})) =$$

$$(-1)^2\alpha^{i_1}\sigma_{2m}((e, \alpha^{i_2}, \dots, \alpha^{\sum_{j=2}^{2m+1} i_j})) + \sum_{k=1}^{2m+2} (-1)^{k+1}\sigma_{2m}((e, (\alpha^{\sum_{j=1}^{l-1} i_j})_{l=2}^{k-1}), (\alpha^{\sum_{j=1}^{l-1} i_j})_{l=k+1}^{2m+2})) =$$

$$\begin{cases} (-1)^{m+2}\alpha^{i_1}x_{2m} & \text{if } \bigwedge_{k=1}^{m}(i_{2k} + i_{2k+1} \geq n) \\ 0 & \text{otherwise} \end{cases} + \lambda_{(i_j)_{j=1}^{2m+1}} \cdot e x_{2m}.$$

By (A.2.3) we have $d_{2m-1}\sigma_{2m}\partial_{2m}(c) = \sigma_{2m-1}\partial_{2m-1}\partial_{2m}(c) = 0$, thus $c'' \in \ker d_{2m-1}$ and finally

$$\lambda_{(i_j)_{j=1}^{2m+1}} = \begin{cases} (-1)^{m+1} & \text{if } \bigwedge_{k=1}^{m}(i_{2k} + i_{2k+1} \geq n), \\ 0 & \text{otherwise.} \end{cases}$$

This shows $c' = c''$ as needed. The analog of (A.2.2) and (A.2.3) for the odd case holds similarly and alternating induction shows $(\sigma_r)_{r=0}^{\infty}$ is a chain map.
Similar calculations cleverly using the properties of chain complexes in alternating induction as above show all other properties of a quasi-isomorphism. ☐

By applying the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(\bullet, A)$ we get complexes whose homology is the group cohomology $H^r(G, A)$ of G with coefficients in a A. To identify $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{r+1}], A)$ with $\mathrm{Map}(G^r, A)$ we use the isomorphisms

$$\begin{aligned} \rho : \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{r+1}], A) \to \mathrm{Map}(G^r, A), \quad f \mapsto \\ \rho(f) : G^r \to A, (g_i)_{i=1}^{r} \mapsto f(e, g_1, g_1g_2, \ \dots \ , g_1g_2\cdots g_r). \end{aligned} \tag{A.2.4}$$

For the standard complex this yields a complex in inhomogeneous form

$$\mathrm{Map}(G^0, A) \xrightarrow{\partial^0} \mathrm{Map}(G^1, A) \xrightarrow{\partial^1} \mathrm{Map}(G^2, A) \xrightarrow{\partial^2} \cdots , \tag{A.2.5}$$

$$(\partial^r f)(g_i)_{i=1}^{r+1} := g_1.(f((g_i)_{i=2}^{r+1})) + \sum_{k=1}^{r}(-1)^k f((g_i)_{i=1}^{k-1}, g_k g_{k+1}, (g_i)_{i=k+2}^{r+1}) + (-1)^{r+1} f((g_i)_{i=1}^{r}).$$

For $G = \langle\alpha\rangle$ cyclic we may use $\rho' : \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \to A, f \mapsto f(e)$ to get from the efficient complex

$$Ax^0 \xrightarrow{d^0} Ax^1 \xrightarrow{d^1} Ax^2 \xrightarrow{d^2} \cdots ,$$

$$d^r(ax^r) := \begin{cases} (a - \alpha.a)x^{r+1} =: (\Delta(a))x^{r+1} & r \text{ even,} \\ (\sum_{i=0}^{n-1} \alpha^i.a)x^{r+1} =: (N(a))x^{r+1} & r \text{ odd.} \end{cases}$$

We call Δ a difference map and N a norm map - they depend on the cyclic group and its given generator. An equation of the form $N(x) = a$, resp. $\Delta(x) = a$, is called a norm equation, resp. a difference equation, where x is the unknown and a is given. Note that $\ker(d^2) = \ker(\Delta) = A^G \subset A$ where A^G denotes the G-invariants of A.

We get a quasi-isomorphism given by $(\tau^r)_{r=0}^{\infty}$ and $(\sigma^r)_{r=0}^{\infty}$, and a homotopy $(h^r)_{r=1}^{\infty}$ from $(\mathrm{id})_{r=0}^{\infty}$ to $(\sigma^r \circ \tau^r)_{r=0}^{\infty}$.

COROLLARY A.5. *Let $G = \langle\alpha\rangle$ be a finite cyclic group of order n and $f \in \mathrm{Map}(G^2, A)$ a 2-cocycle for the G-module A. Via τ^2 we get $a := -\sum_{j=0}^{n-1} f(\alpha^j, \alpha) \in A^G$. Then f is a coboundary if and only if a lies in the image of the norm map. Furthermore if $a = N(a')$ we get a 1-cochain satisfying $\partial^1(f') = f$ which is given by*

$$f'(\alpha^i) := (\sigma^1(a') + h^2(f))(\alpha^i) = f(\alpha^i, e) - \sum_{j=0}^{i-1}(f(\alpha^j, \alpha) + \alpha^j.a').$$

In this way, Problem (A.1) for cyclic groups is reduced to solving norm equations.

A.3. Shapiro's lemma, restriction and corestriction map: from general finite groups to p-groups

This section is mainly based on ideas in [**151**, ch. 2]. For this section fix $r \in \mathbb{N}_0$. We give a solution to the following problem, and use it to reduce Problem (A.1_r) for general G to Problem (A.1_r) for p-groups for the primes p dividing the order of G.

PROBLEM A.6. Given a finite group G, a G-module A and a subgroup $S < G$ such that Problem (A.1_r) is solvable for S denote by $A\,|_S$ the S-module which is the restriction of A. Decide for a given r-cocycle f for the standard complex for A if the restriction of $[f]$ to $H^r(S, A\,|_S)$ vanishes, and if so, exhibit explicitly an $(r-1)$-cochain $f' \in \mathrm{Map}(G^{r-1}, A)$ such that $\partial^{r-1} f' = [G : S]f$, where $[G : S]$ denotes the index of S in G.

To make it easier for the reader to put the following pieces into an organized bigger picture, here is a commutative diagram, on which we elaborate afterwards:

$$\begin{tikzcd}
H^r(G,A) \arrow[r, "\epsilon'^r_A"] \arrow[rr, bend right, "\mathrm{res}'^r"'] \arrow[rrrr, bend left, "m_{[G:S]}"] & H^r(G,\mathrm{Maps_S}(G,A\,|_S)) \arrow[r, "\cong\ \mathrm{sh}'^r"] \arrow[rr, bend left, "\mathrm{id}"] & H^r(S,A\,|_S) \arrow[r, "\cong\ \mathrm{ish}'^r"] \arrow[rr, bend right, "\mathrm{cores}'^r"'] & H^r(G,\mathrm{Maps_S}(G,A\,|_S)) \arrow[r, "\eta'^r_A"] & H^r(G,A).
\end{tikzcd} \tag{A.3.1}$$

These maps depend on S. Later we indicate this by a subscript S. For an S-module B we have a G-module:

$$\mathrm{Maps_S}(G,B) := \{f \in \mathrm{Map}(G,B) : \forall g \in G, s \in S : f(sg) = s.f(g)\};$$
$$\forall g \in G : \forall f \in \mathrm{Maps_S}(G,B) : (g.f) : G \to B, g' \mapsto f(g'g).$$

In the notation of [**151**, p. 27] we have $\pi^*_{S\to G}A = A\,|_S$ and $\pi_{*S\to G}B = \mathrm{Maps_S}(G,B)$, but we will use the more descriptive notation instead. Let $T \subset G$ be a set of right coset representatives for S. According to [**151**, p. 29] we get a monomorphism ϵ_A and an epimorphism η_A of G-modules:

$$\epsilon_A : A \to \mathrm{Maps_S}(G,A\,|_S), a \mapsto f : G \to A\,|_S, g \mapsto g.a;$$
$$\eta_A : \mathrm{Maps_S}(G,A\,|_S) \to A, f \mapsto \sum_{t\in T} t^{-1}.f(t).$$

(Note the use of left cosets in [**151**, p. 34] for contrast.) The map η_A is independent of the choice of coset representatives T. Denoting the multiplication by $[G:S]$-map on the G-module A by $m_{[G:S]}$, we have $\eta_A \circ \epsilon_A = m_{[G:S]}$. The morphisms η_A and ϵ_A induce morphisms on cochain complexes, respectively on cohomology groups, which we will indicate, e.g., by η^r_A, respectively η'^r_A and satisfy

$$\eta^i_A \circ \epsilon^i_A = m_{[G:S]}. \tag{A.3.2}$$

We give the induced morphisms on the complex level explicitly for $i \in \mathbb{N}_0$:

$$\epsilon^i_A : \mathrm{Map}(G^i,A) \to \mathrm{Map}(G^i,\mathrm{Maps_S}(G,A\,|_S)), \big(f : G^i \to A\big) \mapsto$$
$$\big(G^i \to \mathrm{Maps_S}(G,A\,|_S), (g_j)^i_{j=1} \mapsto (G \to A\,|_S, g' \mapsto g'.f((g_j)^i_{j=1}))\big);$$
$$\eta^i_A : \mathrm{Map}(G^i,\mathrm{Maps_S}(G,A\,|_S)) \to \mathrm{Map}(G^i,A)$$
$$\big(f' : G^i \to \mathrm{Maps_S}(G,A\,|_S), (g_j)^i_{j=1} \mapsto f'_{(g_j)^i_{j=1}}\big) \mapsto \big(G^i \to A, (g_j)^i_{j=1} \mapsto \sum_{t\in T} t^{-1}.f'_{(g_j)^i_{j=1}}(t)\big).$$

We explain sh'^r and ish'^r in the following notation. For $\mathfrak{g} := (g', (g_i)^r_{i=1}) \in G^{r+1}$, define $\mathfrak{s} := (s_0, (s_i)^r_{i=1}) \in S^{r+1}$ and $\mathfrak{t} := (t_0, (t_i)^r_{i=1}) \in T^{r+1}$ by the condition below and define $\tilde{s}_i$ and $\tilde{t}_i$ in the same way for $(\tilde{g}', (\tilde{g}_i)^r_{i=1}) \in G^{r+1}$:

$$g' = s_0 t_0 \quad \text{and} \quad \forall 1 \le i \le r : t_{i-1} g_i = s_i t_i.$$

The complexes we use below will all be derived from the standard resolution for G respectively S. The situation is similar to the dual of (A.2.1). The name sh^i is reminiscent of the fact that these morphisms induce the isomorphisms of Shapiro's lemma on the level of cohomology groups and the ish^i induce the "inverse Shapiro isomorphisms", cf. [**151**, p. 31]. For the moment we will work with a general S-module B, though later we will take this to be $A\,|_S$.

PROPOSITION A.7. *Define* $(\mathrm{sh}^i)_{i=0}^{\infty}$, $(\mathrm{ish}^i)_{i=0}^{\infty}$ *and* $(k^i)_{i=1}^{\infty}$ *as follows:*

$$\begin{gathered}
\mathrm{sh}^i : \mathrm{Map}(G^i, \mathrm{Map_S}(G,B)) \to \mathrm{Map}(S^i, B),\\
\big(f : G^i \to \mathrm{Map_S}(G,B), (g_j)_{j=1}^i \mapsto f_{(g_j)_{j=1}^i}\big) \mapsto \big(S^i \to B, (g_j)_{j=1}^i \mapsto f_{(g_j)_{j=1}^i}(e)\big);\\
\mathrm{ish}^i : \mathrm{Map}(S^i, B) \to \mathrm{Map}(G^i, \mathrm{Map_S}(G,B)), \big(f' : S^i \to B\big) \mapsto\\
\big(G^i \to \mathrm{Map_S}(G,B), (g_j)_{j=1}^i \mapsto (G \to B, g' \mapsto s_0.(f'(s_j)_{j=1}^i))\big);\\
k^i : \mathrm{Map}(G^i, \mathrm{Map_S}(G,B)) \to \mathrm{Map}(G^{i-1}, \mathrm{Map_S}(G,B)),\\
\big(f : G^i \to \mathrm{Map_S}(G,B), (g_j)_{j=1}^i \mapsto f_{(g_j)_{j=1}^i}\big) \mapsto\\
\big(G^{i-1} \to \mathrm{Map_S}(G,B), (\tilde g_j)_{j=1}^{i-1} \mapsto (G \to B, \tilde g' \mapsto \sum_{j=0}^{i-1} (-1)^j \tilde s_0.(f_{((\tilde s_l)_{l=1}^j, \tilde t_j, (\tilde g_l)_{l=j+1}^{i-1})}(e)))\big).
\end{gathered}$$

Then $(\mathrm{sh}^i)_{i=0}^{\infty}$ *and* $(\mathrm{ish}^i)_{i=0}^{\infty}$ *are a quasi-isomorphisms via the homotopy* $(k^i)_{i=1}^{\infty}$ *from* $(\mathrm{id})_{i=0}^{\infty}$ *to* $(\mathrm{ish}^i \circ \mathrm{sh}^i)_{i=0}^{\infty}$*:*

$$\mathrm{id} - \mathrm{ish}^i \circ \mathrm{sh}^i = k^{i+1} \circ \partial^i + \partial^{i-1} \circ k^i, \tag{A.3.3}$$

$$\mathrm{id} = \mathrm{sh}^i \circ \mathrm{ish}^i. \tag{A.3.4}$$

PROOF. One checks (A.3.3) and (A.3.4) by straightforward calculations. $\square$

We remark that the definition of the k^i is inspired by [**147**, Lemma III.2.1].

The lower part of diagram (A.3.1) is just mentioned for completeness. The restriction map res'^r and corestriction cores'^r (called transfer in [**151**]) are defined to be the compositions $\mathrm{sh}'^r \circ \epsilon'^r$ and $\eta'^r \circ \mathrm{ish}'^r$ respectively; this is in accordance with [**151**]. Working with the refined factorization $m_{[G:S]} = \eta'^r \circ \mathrm{ish}'^r \circ \mathrm{sh}'^r \circ \epsilon'^r$ rather than the well known $m_{[G:S]} = \mathrm{res}'^r \circ \mathrm{cores}'^r$ allows us to apply the homotopy k^r to get explicit formulas for lifting cocycles.

Keep in mind, that the $\eta^r, \mathrm{ish}^r, \mathrm{sh}^r, \epsilon^r$ form morphisms of complexes, i.e., we have $\eta^r \circ \partial^{r-1} = \partial^{r-1} \circ \eta^{r-1}$, etc.

For a given r-cocycle f of Problem (A.6) we have: $\mathrm{res}'^r([f]) = [\mathrm{sh}^r \circ \epsilon^r(f)]$. By assumption we can check liftability of $\mathrm{sh}^r \circ \epsilon^r(f)$ and if so determine a lift $\tilde f \in \mathrm{Map}(S^{r-1}, A\,|_S)$ such that $\partial^{r-1} \tilde f = \mathrm{sh}^r \circ \epsilon^r(f)$. For such an $\tilde f$ we combine $\partial^r(f) =$

0, (A.3.2) and (A.3.3) to get

$$\partial^{r-1} \circ \eta^{r-1} \circ \operatorname{ish}^{r-1}(\tilde{f}) = \eta^r \circ \operatorname{ish}^r \circ \partial^{r-1}(\tilde{f}) + \eta^r \circ k^{r+1} \circ \epsilon^{r+1} \circ \partial^r(f) =$$
$$\eta^r(\operatorname{ish}^r \circ \operatorname{sh}^r \circ \epsilon^r(f) + k^{r+1} \circ \partial^r \circ \epsilon^r(f)) = \eta^r \circ (\operatorname{ish}^r \circ \operatorname{sh}^r + k^{r+1} \circ \partial^r) \circ \epsilon^r(f) =$$
$$\eta^r \circ (\operatorname{id} - \partial^{r-1} \circ k^r) \circ \epsilon^r(f) = \eta^r \circ \epsilon^r(f) - \eta^r \circ \partial^{r-1} \circ k^r \circ \epsilon^r(f) =$$
$$[G : S]f - \partial^{r-1} \circ \eta^{r-1} \circ k^r \circ \epsilon^r(f)$$

and so

$$f' := \eta^{r-1}(\operatorname{ish}^{r-1}(\tilde{f}) + k^r(\epsilon^r(f)))$$

solves Problem (A.6). Summarizing, we have for $r = 2$:

PROPOSITION A.8. *Let G be a finite group, $S < G$ a subgroup with a set of right coset representatives T and $f \in \operatorname{Map}(G^2, A)$ a 2-cocycle. Let $\operatorname{res}^2_S(f) : S^2 \to A\,|_S, (s_1, s_2) \mapsto f(s_1, s_2)$ denote the restricted 2-cocycle for S. Define auxiliary functions $\mathfrak{s} : G \to S$ and $\mathfrak{t} : G \to T$ by requiring $\mathfrak{s}(g)\mathfrak{t}(g) = g$. Then if $\operatorname{res}^2_S(f)$ is a 2-coboundary such that $\partial \tilde{f} = \operatorname{res}^2_S(f)$ for $\tilde{f} \in \operatorname{Map}(S, A\,|_S)$ we get a 1-cochain f' satisfying $\partial^1(f') = [G : S]f$, defined by:*

$$f' : G \to A, g \mapsto \sum_{t \in T} t^{-1}.(\tilde{f}(\mathfrak{s}(tg)) + f(t, g) - f(\mathfrak{s}(tg), \mathfrak{t}(tg))).$$

COROLLARY A.9. *Let G be a finite group, $f \in \operatorname{Map}(G^2, A)$ a 2-cocycle and $|G| = \prod_{i=1}^u p_i^{q_i}$ a prime factorization. We can effectively reduce Problem (A.1) to Problem (A.1) for p_i-Sylow subgroups $S_i < G$, $A\,|_{S_i}$ and restrictions $\operatorname{res}^2_{S_i}(f) \in \operatorname{Map}(S_i^2, A\,|_{S_i})$.*

PROOF. Set $\tilde{p}_i := |G|/p_i^{q_i} = [G : S_i]$ and choose $\hat{p}_1, \ldots, \hat{p}_u \in \mathbb{Z}$ such that $\sum_{i=1}^u \hat{p}_i \tilde{p}_i = 1$ (Chinese Remainder Theorem). If $[f] \neq 0$ then it has order $1 < n||G|$ (cf. **[151**, p. 35]). Assume $p_i|n$ then $[\tilde{p}_i f]$ has order n_i with $p_i|n_i$, in particular $[\tilde{p}_i f] \neq 0$. If all $[\tilde{p}_i f] = 0$ then there are $f_i \in \operatorname{Map}(G, A)$ such that $\partial^1 f_i = \tilde{p}_i f$. Thus for $f' := \sum_{i=1}^u \hat{p}_i f_i'$ we have $\partial^1 f' = \sum_{i=1}^u \hat{p}_i \tilde{p}_i f = f$. Thus $[f] \neq 0 \Leftrightarrow \forall i : [\tilde{p}_i f] = 0$. By (A.8) solving Problem (A.1) for G and $\tilde{p}_i f$ is reduced to solving Problem (A.1) for the S_i and $\operatorname{res}^2_{S_i}(f)$. $\square$

There are obvious analogs for $r \neq 2$. Formulas will get more complicated but are valid when accordingly modified.

A.4. LHS spectral sequence: from solvable groups to cyclic groups

Suppose we have $H \triangleleft G$ a (non-trivial) normal subgroup and set $Q := G/H$. We will denote elements of Q by $q = [g] = gH$, where $g \in G$ is a representative. The Lyndon-Hochschild-Serre (LHS) spectral sequence will be denoted by $E = (E_s^{p,q})$ and is convergent: $E_2^{p,q} = H^p(Q, H^q(H, A)) \Rightarrow H^{p+q}(G, A)$.

By first vertically forming the standard resolution for the G-module A, then taking H-invariants, which yields a complex of Q-modules with action described below, we may resolve each term horizontally and then take Q-invariants. We use the following isomorphism to rewrite the spectral sequence in inhomogeneous form:

$$\operatorname{Map}(Q\times G^i,A)\xrightarrow{\sim}\operatorname{Map}(G^{i+1},A)^H,(f:Q\times G^i\to A)\mapsto$$
$$(G^{i+1}\to A,(g_0,(g_l)_{l=1}^i)\mapsto g_0.f([g_0^{-1}],(g_{l-1}^{-1}g_l)_{l=1}^i)).$$

Note that (A.2.4) is essentially the inverse in the special case $Q=\{1_Q\}$. Some $q\in Q$ acts on $f\in\operatorname{Map}(Q\times G^i,A)$ by multiplying the Q-argument of f from the right by q before applying f. This defines a Q-action.

We get E_0 as in (A.4.1). The differentials are induced by the differentials of the standard resolutions and are given explicitly by (the $(-1)^i$-factor in the $d_0^{j,i}$ is added to ensure anticommutativity of the bicomplex):

$$\partial_0^{j,i}:\operatorname{Map}(Q^j,\operatorname{Map}(Q\times G^i,A))\to\operatorname{Map}(Q^j,\operatorname{Map}(Q\times G^{i+1},A)),$$
$$(f:Q^j\to\operatorname{Map}(Q\times G^i,A),([g_l])_{l=1}^j\mapsto f_{([g_l])_{l=1}^j})\mapsto$$
$$(Q^j\to\operatorname{Map}(Q\times G^{i+1},A),([g_l])_{l=1}^j\mapsto(Q\times G^{i+1}\to A,([\tilde g_0],(\tilde g_{l'})_{l'=1}^{i+1})\mapsto$$
$$(\tilde g_1.f_{([g_l])_{l=1}^j}([\tilde g_0^{-1}\tilde g_1],(\tilde g_{l'})_{l'=2}^{i+1})+\sum_{n=1}^{i}(-1)^n f_{([g_l])_{l=1}^j}([\tilde g_0^{-1}],(\tilde g_{l'})_{l'=1}^{n-1},\tilde g_n\tilde g_{n+1},(\tilde g_{l'})_{l'=n+2}^{i+1})+$$
$$(-1)^n f_{([g_l])_{l=1}^j}([\tilde g_0^{-1}],(\tilde g_{l'})_{l'=1}^{i}))));$$
$$d_0^{j,i}:\operatorname{Map}(Q^j,\operatorname{Map}(Q\times G^i,A))\to\operatorname{Map}(Q^{j+1},\operatorname{Map}(Q\times G^i,A)),$$
$$(f:Q^j\to\operatorname{Map}(Q\times G^i,A),([g_l])_{l=1}^j\mapsto f_{([g_l])_{l=1}^j})\mapsto(Q^{j+1}\to\operatorname{Map}(Q\times G^i,A),([g_l])_{l=1}^{j+1}\mapsto$$
$$(-1)^i([g_1].f_{([g_l])_{l=2}^{j+1}}+(\sum_{n=1}^{j}(-1)^n f_{(([g_l])_{l=1}^{n-1},[g_n][g_{n+1}],([g_l])_{l=n+2}^{j+1})})+(-1)^{j+1}f_{([g_l])_{l=1}^j}));$$

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\operatorname{Map}(Q^0,\operatorname{Map}(Q\times G^2,A)) \arrow[u, "\partial_0^{0,2}"] \arrow[r, "d_0^{0,2}"] & \operatorname{Map}(Q^1,\operatorname{Map}(Q\times G^2,A)) \arrow[u, "\partial_0^{1,2}"] \arrow[r, "d_0^{1,2}"] & \operatorname{Map}(Q^2,\operatorname{Map}(Q\times G^2,A)) \arrow[u, "\partial_0^{2,2}"] \arrow[r, "d_0^{2,2}"] & \cdots \\
\operatorname{Map}(Q^0,\operatorname{Map}(Q\times G^1,A)) \arrow[u, "\partial_0^{0,1}"] \arrow[r, "d_0^{0,1}"] & \operatorname{Map}(Q^1,\operatorname{Map}(Q\times G^1,A)) \arrow[u, "\partial_0^{1,1}"] \arrow[r, "d_0^{1,1}"] & \operatorname{Map}(Q^2,\operatorname{Map}(Q\times G^1,A)) \arrow[u, "\partial_0^{2,1}"] \arrow[r, "d_0^{2,1}"] & \cdots \\
\operatorname{Map}(Q^0,\operatorname{Map}(Q\times G^0,A)) \arrow[u, "\partial_0^{0,0}"] \arrow[r, "d_0^{0,0}"] & \operatorname{Map}(Q^1,\operatorname{Map}(Q\times G^0,A)) \arrow[u, "\partial_0^{1,0}"] \arrow[r, "d_0^{1,0}"] & \operatorname{Map}(Q^2,\operatorname{Map}(Q\times G^0,A)) \arrow[u, "\partial_0^{2,0}"] \arrow[r, "d_0^{2,0}"] & \cdots
\end{tikzcd} \tag{A.4.1}$$

We show $\ker(d_0^{j+1,i})=\operatorname{im}(d_0^{j,i})$ for fixed i, i.e., the $\operatorname{Map}(Q\times G^i,A)$ are acyclic Q-modules. Let $\pi^{j,i}:\operatorname{Map}(Q^j,\operatorname{Map}(Q\times G^i,A))\xrightarrow{\sim}\operatorname{Map}(Q^j,\operatorname{Map}(Q,A))^{|G^i|}$ define the obvious isomorphism of complexes of abelian groups in j. We are left to show that $\operatorname{Map}(Q^j,\operatorname{Map}(Q,A))$ with differentials ∂^j similar to those of (A.2.5) is exact. We call an $f\in\operatorname{Map}(Q^j,\operatorname{Map}(Q,A))\cong\operatorname{Map}(Q^j\times Q,A)$ constant, if it is constant in both arguments. Easy calculations show that a constant $f \neq 0$ is eaxctly in both $\ker(\partial^{j+1})$ and $\mathrm{im}(\partial^j)$ if j is even and in neither if j is odd. We apply (A.3.3) for $\{1_Q\} < Q$ and $A|_{\{1_Q\}}$. Note that $\mathrm{ish}^j \circ \mathrm{sh}^j$ in this case is an idempotent projection on the constant maps. These facts about constant f and ∂^j being a differential leave us with showing $\ker(\partial^{j+1}) \cap \ker(\mathrm{ish}^i \circ \mathrm{sh}^i) \subset \mathrm{im}(\partial^j)$. By (A.3.3) we have for an $f \in \ker(\partial^{j+1}) \cap \ker(\mathrm{ish}^i \circ \mathrm{sh}^i)$, that

$$f = (\mathrm{id} - \mathrm{ish}^{j+1} \circ \mathrm{sh}^{j+1} - k^{j+2} \circ \partial^{j+1})(f) = (\partial^j \circ k^{j+1})(f) \in \mathrm{im}(\partial^j).$$

Explicitly for $i = j = 0$ we get for an $f : Q^1 \to \mathrm{Map}(Q \times G^0, A), q \mapsto f_{(q)}$ satisfying $d_0^{1,0}(f) = 0$ some

$$\tilde{f} : Q^0 \to \mathrm{Map}(Q \times G^0, A), () \mapsto (Q \to A, q \mapsto f_q(e))$$

such that $d_0^{0,0}(\tilde{f}) = f$. This induces a morphism $\phi : \ker(d_0^{1,0}) \xrightarrow{\sim} \mathrm{im}(d_0^{0,0}) \hookrightarrow \mathrm{Map}(Q^0, \mathrm{Map}(Q \times G^0, A))$.

The limiting term of the spectral sequence is the cohomology of the associated total complex. Since the $\mathrm{Map}(Q \times G^i, A) \cong \mathrm{Map}(G^{i+1}, A)^H$ are acyclic the total complex is quasi-isomorphic to the complex given by the abelian groups $(\mathrm{Map}(G^{i+1}, A)^H)^Q = \mathrm{Map}(G^{i+1}, A)^G \cong \mathrm{Map}(G^i, A)$, i.e., to the complex (A.2.5) that computes $H^*(G, A)$. Therefore we can represent a cohomology class $[f] \in H^p(G, A)$ as follows:

$$\mathrm{Map}(G^p, A) \to \bigoplus_{j+i=p} \mathrm{Map}(Q^j, \mathrm{Map}(Q \times G^i, A)), \quad (f : G^p \to A) \mapsto$$
$$\Big(\quad (f' : Q^0 \to \mathrm{Map}(Q \times G^p, A), () \mapsto (Q \times G^p \to A, (q, (g_l)_{l=1}^p) \mapsto f(g_l)_{l=1}^p)), \quad (0)_{l=1}^p \Big).$$

This map gives an isomorphism onto the subgroup $\ker(d_0^{0,p}) \oplus \bigoplus_{j+i=p, j\neq 0} 0$. Call this isomorphism ψ^p. We define

$$\chi^i : \mathrm{Map}(Q \times G^i, A) \xrightarrow{\sim} \mathrm{Map}(G^i, \mathrm{Map}_H(G, A|_H)), \quad (f : Q \times G^i \to A) \mapsto$$
$$(G^i \to \mathrm{Map}_H(G, A|_H), (g_l)_{l=1}^i \mapsto (G \to A|_H, g' \mapsto g'.(f([g'], (g_l)_{l=1}^i)))).$$

PROPOSITION A.10. *Let G be a finite group, A a G-module, $H \triangleleft G$ a normal subgroup and $Q := G/H$ the quotient group. Assume that:*

(1) *we have an algorithm to solve Problem (A.1) for the groups H, Q and the modules $A|_H$ respectively A^H,*
(2) *we have $H^1(H, A|_H) = 0$ effectively, i.e. an algorithm for producing for a given 1-cocycle a lifting 0-cochain.*

Then we can solve Problem (A.1) for G and A.

PROOF. We give explicit formulas and leave checking to the reader.

Let $f \in \mathrm{Map}(G^2, A)$ be a 2-cocycle as above. Define $\alpha \in \mathrm{Map}(Q \times G^2, A)$ by $(\alpha, 0, 0) = \psi^2(f)$. We have $\partial_0^{0,2}(\alpha) = 0$ as well as $d_0^{0,2}(\alpha) = 0$. A homomorphism $H^2(G, A) \to H^2(H, A|_H)^Q$ is given on the cocycle level by $f \mapsto \mathrm{sh}^2(\chi^2(\alpha_{()}))$ where

sh^2 is as in Proposition A.7 for $S = H < G$ and $B = A\,|_H$. A necessary condition for $[f] = 0$ is $[\mathrm{sh}^2(\chi^2(\alpha_{()}))] = 0$. This can be tested effectively by assumption 1 and if so we get a 1-cochain $\tilde{\delta}' : H \to A$ such that $\partial^1(\tilde{\beta}') = \mathrm{sh}^2(\chi^2(\alpha_{()}))$. Define

$$\beta' : Q^0 \to \mathrm{Map}(Q \times G^1, A), () \mapsto (\chi^1)^{-1}(k^2(\chi^2(\alpha_{()})) + \mathrm{ish}^1(\tilde{\beta}')),$$

and $\gamma' := d_0^{0,1}(\beta')$. We have $d_0^{1,1}(\gamma') = 0$. Using (A.3.3) $\partial_0^{0,2}(\alpha) = 0$ implies $\partial_0^{0,1}(\beta') = \alpha$ and $d_0^{0,2}(\alpha) = 0$ implies $\partial_0^{1,1}(\gamma') = 0$. By assumption 2 we get $\tilde{\delta}''_{(q)} \in \mathrm{Map}(H^0, A\,|_H)$ satisfying $\mathrm{sh}^1(\chi^1(\gamma'_{(q)})) = \partial^0(\tilde{\delta}''_{(q)})$ for $q \in Q$. Set

$$\delta''_{(q)} := (\chi^0)^{-1}(k^1(\chi^1(\gamma'_{(q)})) + \mathrm{ish}^0(\tilde{\delta}''_{(q)}))$$

and define $\epsilon'' := d_0^{1,0}(\delta'')$. We get $\partial_0^{1,0}(\delta'') = \gamma'$, $\partial_0^{2,0}(\epsilon'') = 0$ and $d_0^{2,0}(\epsilon'') = 0$. By the last two assertions the $\epsilon''_{(q_1,q_2)} \in \mathrm{Map}(Q \times G^0, A)$ take values only in A^H and are constant maps. Setting $\tilde{\epsilon}''_{(q_1,q_2)} := \epsilon''_{(q_1,q_2)}(e) \in A^H$ we get $\tilde{\epsilon}'' \in \mathrm{Map}(Q^2, A^H)$. For the total complex $T := ((\mathrm{Tot}\, E_0^{\bullet,\bullet})_n, D^n)$ we get isomorphisms $\pi^n : H^n(G, A) \xrightarrow{\sim} H^n(T)$. By construction $D^1(\beta', -\delta'') = (\alpha, 0, 0) - (0, 0, \epsilon'')$ thus $\pi^2([f]) = [(\alpha, 0, 0)] = [(0, 0, \epsilon'')]$. The subgroup of classes of the form $[(0, 0, *)]$ is $E_\infty^{2,0}$. We have by assumption 2 that $E_2^{0,1} \cong H^1(H, A\,|_H)^Q = 0$. This together with LHS being first quadrant implies $d_r^{2-r,r-1} = 0 = d_r^{2,0}$ for $r \geq 2$, i.e., $E_\infty^{2,0} \cong E_2^{2,0} \cong H^2(Q, A^H)$. We get $[f] = 0$ implies $[\tilde{\epsilon}''] = 0 \in H^2(Q, A^H)$. The later can be tested effectively by assumption 1 and if so we get some $\tilde{\delta}''' \in \mathrm{Map}(Q^1, A^H)$ which lifts $\tilde{\epsilon}''$. We can regard $\tilde{\delta}'''$ as a

$$\delta''' \in \mathrm{Map}(Q^1, \mathrm{Map}(Q \times G^0, A))$$

such that $d_0^{1,0}(\delta''') = \epsilon''$ and $\partial_0^{1,0}(\delta''') = 0$. In summary either we have found $[f] \neq 0$ or

$$\beta := \beta' - \partial_0^{0,0}\phi(-\delta'' + \delta''') \in \ker(d_0^{0,1})$$

satisfies $\partial_0^{0,1}(\beta) = \alpha$ and $d_0^{0,1}(\beta) = 0$. So we may apply the inverse of ψ^1 to get a 1-cochain

$$f' \in \mathrm{Map}(G^1, A)$$

that finally satisfies $\partial^1(f') = f$. Since every step is explicitly given this shows the effectivity of the procedure. ☐

Corollary A.11. *Let G be the Galois group of a solvable finite field extension K/k and $A := K^* = K \setminus \{0\}$ the usual G-module. Then we can effectively reduce solving Problem (A.1) for G and A to solving Problem (A.1) for cyclic groups and associated group modules which arise from intermediate field extensions of K/k.*

Proof. Since the statement is clearly true for cyclic groups we may use induction on the size of the Galois group.

As G is solvable it has a normal subgroup H with abelian quotient Q and we may assume Q to be non-trivial cyclic. To H corresponds the Galois extension K/K^H, and $A\,|_H$ is still K^*. To Q corresponds the cyclic Galois extension K^H/k and $A^H = (K^H)^*$.

By induction assumption 1 of Proposition (A.10) is satisfied and assumption 2 is satisfied by Hilbert's Theorem 90, and usual proofs (e.g., [**151**, p. 94]) give $H^1(H, A\,|_H) = 0$ effectively. The above proposition now establishes the corollary. □

REMARK A.12. In Corollary (A.11) one could replace cyclic by cyclic of prime order. Proposition (A.10) generalizes to higher $H^r(G, A)$ assuming effective vanishing of $H^i(H, A\,|_H)$ for $0 < i < r$.

A.5. Applications

First, we apply the previous results and prove Theorem A.2:

PROOF. Given $[f] \in H^2(G, A)$ we can reduce to solving Problem (A.1) for a finite collection of p-Sylow subgroups and their associated Galois extensions by Corollary (A.9). Since p-groups are solvable we reduce further to finitely many intermediate cyclic Galois extensions by Corollary (A.11). Finally Corollary (A.5) reduces this to solving cyclic norm equations. □

REMARK A.13. (i) Remark (A.12) shows that we can assume the cyclic intermediate extensions to be of prime order.
(ii) The results stated in Corollary (A.9) and Corollary (A.5) generalize well to arbitrary group cohomology for finite groups and also arbitrary degree as indicated. Proposition (A.10) generalizes also with respect to both aspects but it requires rather strong effective vanishing conditions, and Corollary (A.11) generalizes as well under effective vanishing conditions. Consequently Theorem (A.2) generalizes in this situation also to higher cohomology groups.
(iii) For C_1-fields, norm equations are always solvable for intermediate extensions as in the theorem and all higher cohomology vanishes. For instance, Tsen's Theorem tells us that this is the case for function fields in one variable over an algebraically closed field. In this case an effective algorithm for solving such norm equations is well known and described, e.g., in [**149**, Thm. C.4.2]. So the effective methods of this paper give an algorithm for solving Problem (A.1) for this class of fields. Since difference equations can be effectively solved by Hilbert's Theorem 90 we have using (ii) inductively effective vanishing of all higher cohomology.
(iv) For number fields, cyclic norm equations are solvable if and only if there are no local obstructions by the Hasse Norm Theorem [**143**]. When a solution exists there are effective algorithms to produce a solution, e.g., [**141**], and these have been implemented for instance in the computer algebra program `MAGMA`. See Remark A.14 for further comments on this case.

We give a procedure to evaluate local invariants [**150**, XIII, §3] for number fields as stated in Proposition A.3:

PROOF. Let k_v be the completion of k at v. Since K/k is Galois there is only one place w in K extending v up to Galois conjugacy; denote the completion of

K for w by K_w. Define $Q_1 := \mathrm{Gal}(K_w/k_v) < \Gamma$ and let $\hat{f} \in \mathrm{Map}(Q_1^2, K_w^*)$ be the induced 2-cocycle.
For v an infinite place the Brauer group is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and being able to determine vanishing of a class suffices.
If v is finite let $d||Q_1|$ satisfy $d[\hat{f}] = 0$, e.g., $d = |Q_1|$ (cf. [**151**, p. 35]). Let $k_{v,d}/k_v$ be the up to isomorphism unique unramified extension of degree d effectively constructable as a cyclotomic extension, v_d its valuation, $A' := k_{v,d} \cdot K_w$ a composition of fields and $A := A'^*$.

$$\begin{tikzcd}
& A' \arrow[dl, dash, "H_2"'] \arrow[d, dash, "H_1"] \arrow[ddl, dash, "G"] \\
k_{v,d} \arrow[d, dash, "Q_2\cong G/H_2"'] & K_w \arrow[dl, dash, "Q_1\cong G/H_1"] \\
k_v &
\end{tikzcd}
\qquad
\begin{tikzcd}
& H^2(Q_2, A^{H_2}) \arrow[d, "\mathrm{inf}_2"] & \\
H^2(Q_1, A^{H_1}) \arrow[r, "\mathrm{inf}_1"] & H^2(G, A) \arrow[r, "\mathrm{res}_1"] \arrow[d, "\mathrm{res}_2"] & H^2(H_1, A\,|_{H_1})^{Q_1} \\
& H^2(H_2, A\,|_{H_2})^{Q_2} &
\end{tikzcd}$$

We get Galois extensions as indicated in the diagram with the corresponding Galois groups. By Hilbert's Theorem 90 all relevant H^1 vanish and by [**151**, p. 50] we get for each of the two field extension towers higher 5-term sequences, of which the first 3 terms of each constitute the diagram on the right.
By [**150**, XIII, §3] we have $0 = d\,\mathrm{inv}_v(\mathrm{inf}_1[\hat{f}]) = \mathrm{inv}_v(\mathrm{res}_2(\mathrm{inf}_1[\hat{f}]))$, thus $\mathrm{res}_2(\mathrm{inf}_1[\hat{f}]) = 0$. As in the proof of Proposition (A.10) we get effectively a 2-cocycle $\epsilon'' \in \mathrm{Map}(Q_2^2, A^{H_2})$ with $\mathrm{inf}_2[\epsilon''] = \mathrm{inf}_1[\hat{f}]$. For this we need to lift a 2-coboundary in $\mathrm{Map}(H_2^2, A\,|_{H_2})$ to a 1-cochain in $\mathrm{Map}(H_2^1, A\,|_{H_2})$ which by Theorem A.2 and Remark A.13(i) is reduced to solving norm equations for cyclic extensions of local fields of prime degree. This task is effectively solvable by computations over finite residue fields (cf. [**140**, III.1]). The precision needed for solutions of such a norm equation increases only linearly in the precision of the right hand side – bounds can be read off from loc. cit. We get

$$\mathrm{inv}_v[f] = \mathrm{inv}_v[\epsilon''] = \frac{1}{d} v_d(\tau^2(\epsilon'')^{-1}) = \frac{1}{d} v_d(\sum_{j=0}^{d-1} \epsilon''(\alpha^j, \alpha)),$$

where α is the element of Q_2 inducing the Frobenius on the residue field, τ^2 as in section A.2 maps into $k_{v,d}^*$. $\square$

Remark A.14. Fieker in [**142**] solves Problem (A.1) for number fields K/k by identifying a finite set of "critical" primes S_{dir} such that lifting a 2-coboundary with coefficients in K^* is reduced to lifting with coefficients in the associated S-units $U_{S,K}$. After representing $U_{S,K}$ as finitely generated abelian group an algorithm of Holt (cf. [**144**]) is used to find a lifting for $U_{S,K}$-coefficients. Reducing Problem (A.1) directly to S-unit equations was previously outlined in [**136**, Thm. 2/3] both over number fields and function fields. The choice of the set of critical primes differs slightly.

In our approach we reduce to a finite collection of cyclic norm equations. For number fields these can be treated by S-unit equations (cf. [**139**, 7.5]) and we get a collection $\{S_{cyc,i}\}$ of critical primes for intermediate field extensions. In both cases one needs to determine class groups and unit groups. While S_{dir} needs to contain enough primes to generate the whole class group the $S_{cyc,i}$ need only generate parts of the relative pseudo class groups (cf. [**139**, Corollary 7.5.7]). One method to determine class groups (and units) uses induction on relative pseudo class groups (cf. [**139**, 7.3.3]), so this information is no harder to obtain as the absolute class group, except that we need to obtain sequences of relative pseudo class groups for each prime number dividing the degree separately.

In both approaches the critical sets include the support of the 2-cocycle in question. In particular in our iterative process this support might grow in size, however Fieker already gives heuristics to counter this in [**142**, 7.].

We outline two modifications of our approach. We may first compute a normal series decomposition of G consisting only of simple and cyclic factors (not necessarily of prime order) and apply Corollary A.9 only for non-cyclic simple factors. This should reduce the amount of norm equations to be solved and thus costly class group computations. Instead of standard resolutions one may use minimal resolutions (cf. [**138**, Thm. 4.3]) to reduce the costs in evaluating the linear maps to relate the solutions to the different norm equations. This needs a dynamic framework to algorithmically create the first terms of diagrams like (A.2.1), but this is standard.

This can, e.g., be applied in the setting of [**145**, Prop. 5 (iv)] for dihedral extensions of degree $2n$. As a dihedral group admits a normal series with 2 cyclic factors only two norm equations need to be solved independent of the number of prime factors of $2n$. In the dihedral case there is also a resolution which is of ranks $1, 2, 3$ in degrees $1, 2, 3$ (cf. [**145**, Prop. 5 (iv)]) compared to ranks $1, 2n, (2n)^2$ for the standard resolution. Depending on the size of the relative pseudo class group which might complicate solving a single norm equation it might nevertheless be advisable to split the degree n norm equation in several parts, if possible.

The considerations above are for the particular case of number fields. To compute local invariants as in Proposition A.3 we work over local fields and their finite Galois extension are always solvable. Neither do the costly S-unit computations occur, nor several towers of field extensions over already large base fields for each prime divisor of the degree of the extension. And the norm equations can be solved by computations over finite fields.

APPENDIX B

Implementations of Algorithms

B.1. Finding Rational Solutions

The following code searches for primitive, integral solutions $[x_0 : x_1 : x_2 : x_3]$ of $x_0^4 + 3x_1^4 = 4x_2^4 + 9x_3^4$ such that the value of $x_0^4 + 3x_1^4$ is between two given bounds a, b. The code is specifically tailored for the diagonal quartic given by this equation, but can easily be modified for other diagonal quartics.

```
/*We need the integers for coercion in gcd-computations like
  "Gcd(Z!(4/2),Z!(10/2));".
*/
Z:=Integers();

/*This function compares two quadruples according to their value
  of x0^4+3x1^4=4x2^4+9x3^4. This is for sorting purposses to make a
  nicer representation of solutions.
*/
quadcomp:=function(c,d)
n:=c[1]^4+3*c[2]^4;
m:=d[1]^4+3*d[2]^4;
if n eq m
  then if c[1] eq d[1]
    then return c[3]-d[3];
    else return c[1]-d[1];
  end if;
  else return n-m;
end if;
end function;

/*In this procedure we set up a list of residues which occur both on the
  left and on the right of x0^4+3x1^4=4x2^4+9x3^4. This list is used to
  discard many impossible candidates from the start.
*/
pairs:=[[2,8],[3,3],[5,2],[7,2],[13,2]];
singles:=[q[1]^q[2]: q in pairs];
residues:=[];
for q in singles do
```

```
  r:=[0 .. q];
  rr:=[[a,b]: a in r, b in r];
  l1:={}; l2:={};
  for c in rr do
    val1:=c[1]^4+3*c[2]^4;
    val2:=4*c[1]^4+9*c[2]^4;
    Include(~l1,val1 mod q);
    Include(~l2,val2 mod q);
  end for;
  print "Modulo ",q," there are ",#l1," classes for x0^4+3x1^4, ",#l2,
    " classes for 4x2^4+9x3^4 and ",#(l1 meet l2)," classes in the
    intersection";
  Append(~residues,(l1 meet l2));
end for;

/*This function searches for solutions with a <= x0^4+3x1^4 <= b. First
  all potential pairs [x0,x1] are tested against the residue list and if
  they pass the x0^4+3x1^4 is put into the set "leftCandValues" which in
  general is faster to test containment for than for a list and
  [x0,x1,x0^4+3x1^4] stored separately in "leftCand". Then all potential
  pairs [x2,x3] are tested against the residue list first to avoid costly
  look ups in sets and then the 4x2^4+9x3^4 is tested for containment in
  the set "leftCandValues". After passing this test [x2,x3,4x2^4+9x3^4] is
  stored in "rightPairs" and a definitive set of values "vals" occuring for
  both expressions generated. Then the values of x0^4+3x1^4 of points in
  the list "leftCand" are compared to the "vals" and a definitive list
  "leftPairs" constructed. Finally the pairs in "leftPairs" are matched to
  the pairs in "rightpairs" and a list of rational points is generated.
  This list is returned and a little statistic printed.
*/
searchbetween:=function(a,b)
range0:=[0..Floor(Root(b,4))];
leftCandValues:={};
leftCand:=[];
for x0 in range0 do
  lower:=Ceiling((a-x0^4)/3);lower:=Ceiling(Root(Maximum(0,lower),4));
  uper:=Floor(Root(((b-x0^4)/3),4));
  for x1 in [ lower .. uper] do
    val1:=x0^4+3*x1^4;
    if (((val1 mod singles[1]) in residues[1]) and
      ((val1 mod singles[2]) in residues[2]) and
      ((val1 mod singles[3]) in residues[3]) and
      ((val1 mod singles[4]) in residues[4])) then
```

```
      Include(~leftCandValues,x0^4+3*x1^4);
      Append(~leftCand,[x0,x1,x0^4+3*x1^4]);
    end if;
  end for;
end for;
vals:={};
rightPairs:=[];
range2:=[0..Floor(Root(b/4,4))];
for x2 in range2 do
  lower:=Ceiling((a-4*x2^4)/9);lower:=Ceiling(Root(Maximum(0,lower),4));
  uper:=Floor(Root(((b-4*x2^4)/9),4));
  for x3 in [ lower .. uper] do
    val2:=4*x2^4+9*x3^4;
    if (((val2 mod singles[1]) in residues[1]) and
      ((val2 mod singles[2]) in residues[2]) and
      ((val2 mod singles[5]) in residues[5])) then
      if val2 in leftCandValues then
        Include(~vals,val2); Append(~rightPairs,[x2,x3,val2]);
      end if;
    end if;
  end for;
end for;
leftPairs:=[];
for c in leftCand do
  if c[3] in vals then Append(~leftPairs,c); end if;
end for;
ratptList:=[];
for pairL in leftPairs do
  for pairR in rightPairs do
    if (pairL[3] eq pairR[3]) then
      Append(~ratptList,[pairL[1],pairL[2],pairR[1],pairR[2]]);
    end if;
  end for;
end for;
print "#{left candidates}: ", #leftCandValues, "#{right pairs}: ",
  #rightPairs, "#{left pairs}: ", #leftPairs,
  "#{nonprimitive rational points in range}: ", #ratptList;
return ratptList;
end function;

/*This function is wrapper function for testbetween. To lower memory
  usage the search region of the potential values is split up into
  smaller intervalls and passed to searchbetween. After each call the
```

```
  rational points are found so far are combined with those found now,
  made primitive and multiples reduced to a single point in the list.
  At the end of each call a statistic about the latest call and the
  complete list of primitive rational points is printed. It is possible
  to give an initialization for the list of rational points. Their
  validity is not checked.
*/
searchRatPt:=function(a,b,init)
complList:=init;
boundlist:=[a];
l:=Ceiling(Root(a/4,4)/100);
u:=Floor(Root(b/4,4)/100);
for i in [l..u] do
  Append(~boundlist,4*(100*i)^4);
end for;
Append(~boundlist,b);
t:=Cputime();
for i in [1..(#boundlist-1)] do
  interlist:=searchbetween(boundlist[i],boundlist[i+1]);
  complList:=complList cat interlist;
  complSet:={};
  for j in [1 .. #complList] do
    pt:=complList[j];
    d:=Gcd([Z!pt[k]: k in [1,2,3,4]]);
    Include(~complSet,[pt[k]/d: k in [1,2,3,4]]);
  end for;
  complListb:=[];
  for pt in complSet do
    Append(~complListb,pt);
  end for;
  complList:=Sort(complListb,quadcomp);
  print i,"-th intervall, evolved time: ", Cputime(t),
    " for bounds from ", boundlist[i], " to ", boundlist[i+1];
  print complList;
  t:=Cputime();
end for;
return complList;
end function;

/*Avoid non-positive numbers to be in the range otherwise Gcd([0,0,0,0])
  would cause a division by 0 and taking 4th roots is not possible.
*/
searchRatPt(1,10^12,[[1,1,1,0],[7,3,5,2]]);
```

B.2. Computing the $\pi_1^{an}(A^{an})$-action on $\mathrm{Pic}_{Y_i/A}[3]$

This is an implementation of the path lifting method to compute the monodromy action. See section 4.3 for remarks on the underlying theory.

```
/*The following three function provide the functionality for lifting
  paths of a base to a cover given by rational functions.
  "liftpaths" takes paths on the base in the format of several lists of
  coordinate tuples which therefor has to be a threefold iterated list
  together with a list of names for the paths which essentially is a
  sequence of numbers reflecting the history of previous lifts. It takes
  as well the polynomial that describes the cover and the ring in which
  it resides. initialize specifies how the different paths begin in the
  same base point. It then splits this data into data for single paths,
  passes it on to "liftpath" and merges the results with previously
  lifted paths.
  "liftpath" takes essentially the input of "liftpaths" but for a single
  path. It first does some initialization and then computes all roots of
  all points in the path. Then it computes the minimum distance between
  all roots laying over a single point and stores it in a list. In the
  next step the the roots for two consequtive points called old and new
  point in the path a passed to "match".
  This function matches closest new roots to a single old root and
  returns the matching in an ordered list corresponding to the order of
  the old roots as well as distance. "match" is called for each old root.
  It might happen that a single new root is closest to two different old
  roots. This is not detected or avoided since it will simply destroy
  the property of being a permutation. When trying to put a permutation
  group structure on this data it will return an error which is a
  valuable indication of the stepsize being too big.
  Then "liftpath" continues the path with the point specified by "match"
  and performs a further numerical consistency check: if a returned
  distance between old and new root is bigger than a 10th of the minimal
  distance between all old roots it prints a warning and some indication
  about the location of this potentially numerically instable situation.
  Then it returns the found paths lifting the single given path and
  appends a number to the path description indicating the order in which
  MAGMA put the roots for the initial point of the path in the base.
  Multiple roots can not be handeled but in the general situation they
  do not occur. If they occur or "almost" occur they are however a sign
  of numerical instability and will produce an error similar to that
  discussed for "match" and will indicate refinement of the step size or
  rerouting of the path around the exceptional discriminant loci.
*/
```

```
match:=function(old,listcandnew)
newindex:=1;
mindist:=Abs(old-listcandnew[1][1]);
for d:=2 to #listcandnew do
  actdist:=Abs(old-listcandnew[d][1]);
  if actdist le mindist
  then newindex:=d; mindist:=actdist;
  end if;
end for;
return listcandnew[newindex][1],mindist;
end function;

liftpath:=function(path,pathdesc,f,PolU,pathinitialize,nmbpaths)
firstf:=PolU!Evaluate(f,path[1]);
if #pathinitialize eq 0 then
  firstroots:=[Roots( firstf )[d][1]: d in [1 .. Degree(firstf)]];
else
  firstroots:=pathinitialize[Degree(firstf)*(nmbpaths-1)+1 ..
    Degree(firstf)*nmbpaths];
end if;
newpathsdesc:=[Append(pathdesc,d): d in [1 .. Degree(firstf)]];
newrootslists:=[Roots( PolU!Evaluate(f,path[k]) ): k in [2 .. #path]];
newpaths:=[];
mindistlist:=[];
for k:=2 to #path do
  mindistlist[k-1]:=Abs(newrootslists[k-1][1][1] -
    newrootslists[k-1][2][1]);
  for d:=1 to Degree(firstf)-1 do
    for D:=d+1 to Degree(firstf) do
      actdist:=Abs(newrootslists[k-1][d][1]-newrootslists[k-1][D][1]);
      if actdist le mindistlist[k-1]
        then mindistlist[k-1]:=actdist;
      end if;
    end for;
  end for;
end for;
for d:=1 to Degree(firstf) do
  newpath:=[];
  newpath[1]:=Append(path[1], firstroots[d]);
  for k:=2 to #path do
    oldroot:=newpath[k-1][#newpath[k-1]];
    newrootslist:=newrootslists[k-1];
    newpoint,dist:=match(oldroot,newrootslist);
```

```
    if 10*dist gt mindistlist[k-1] then
      print "step size to big!", 10*dist, mindistlist[k-1],
        10*dist/mindistlist[k-1], k, newpathsdesc[d];
    end if;
    newpath[k]:=Append(path[k],newpoint);
  end for;
  Append(~newpaths,newpath);
end for;
return newpaths,newpathsdesc;
end function;

liftpaths:=function(paths,pathsdesc,f,PolU,initialize)
newpaths:=[];
newpathsdesc:=[];
for l:=1 to #paths do
  liftedpaths,liftedpathsdesc:=
    liftpath(paths[l],pathsdesc[l],f,PolU,initialize,l);
  Insert(~newpaths,#newpaths+1,#newpaths+1,liftedpaths);
  Insert(~newpathsdesc,#newpathsdesc+1,#newpathsdesc+1,liftedpathsdesc);
end for;
return newpaths,newpathsdesc;
end function;

/*"permutationrepresentation" takes data generated by "liftpaths". If
  the paths begins and ends in the same point the lifting will define a
  permutation of roots if the paths avoid patological discrimanant loci
  which a general path does. This structure is exhibited in this
  function and packaged in a permutation group "G" over the set of
  points "gset" in the cover over the starting point of the paths in
  the base. The group is returned together with "gset" and the full
  permutation group "S" of this set and a set of generators "gens" of
  "G" corresponding to the initial paths in the base.
  Errors while running this function may indicate numerical
  instabilities in the path data.
*/
permutationrepresentation:=function(manypaths,manypathsdesc)
gset:={@ manypathsdesc[1][l]: l in [1 .. #manypathsdesc[1]] @};
S:=Sym(gset);gens:=[S|];
for dmany:=1 to #manypaths do
  perm:=[];
  for d:=1 to #manypaths[dmany] do
    for da:=1 to #manypaths[dmany] do
      if manypaths[dmany][d][#(manypaths[dmany][d])]
```

```
        eq manypaths[dmany][da][1] then
        Append(~perm,manypathsdesc[dmany][da]);
      end if;
    end for;
  end for;
  perm;
  Append(~gens,S!perm);
end for;
G:=sub<S|gens>;
return G,S,gset,gens;
end function;



/*We will compute the action of pi_1^{an}(A) on the 3-torsion of
  the Y_i. Due to symmetry Y_1 and Y_2 behave in the same way so we only
  treat Y_0 and Y_1
*/

/*The following computes the 3rd division polynomial for the Jacobian
  JacY0 of Y_0 needed for the path lifting procedure below as well as a
  field of definition of JacY0[3].
*/
Q:=Rationals();
K<t>:=FunctionField(Q);
Pol<x>:=PolynomialRing(K);
JacY0 := EllipticCurve(Pol!x^3-((1+t^4)^2/4)*x);
g:=Pol!DivisionPolynomial(JacY0,3)/3;g;
K1<xi1>:=ext<K|g>;
Pol1<x1>:=PolynomialRing(K1);
f:=Pol1!g/((x1-xi1)*(x1+xi1));
xi2:=-xi1;
K2<eta1>:=ext<K1|Pol1!(x1^2-(xi1^3-((1+t^4)^2/4)*xi1))>;
Pol2<x2>:=PolynomialRing(K2);
K3<eta2>:=ext<K2|Pol2!(x2^2-(xi2^3-((1+t^4)^2/4)*xi2))>;
Pol3<x3>:=PolynomialRing(K3);
JacF0K3 := EllipticCurve(Pol3!x^3-((1+t^4)^2/4)*x);
O:=JacY0K3![0,1,0];P1:=JacY0K3![xi1,eta1,1];P2:=JacY0K3![xi2,eta2,1];
zeta:=WeilPairing(P1,P2,3);
Q;K;K1;K2;K3;Degree(K1);Degree(K2);Degree(K3);zeta^2+zeta+1 eq 0;


/*Next we let MAGMA construct three loops in the form of lists of points
```

```
  in the complex plane around three of the four removed points of P^1
  connect them to the same base points. We specify the equations for the
  two affine coordinates of points in JacY0[3] as well as the the degree
  2 morphism from Y0 to JacY0 and lift the paths to the covers thereby
  specified. The three loops we chose as generators are encirceling the
  three complex points 1,i,-1 counterclockwise.
*/
CFi<i>:=ComplexField(20);
Pol<x>:=PolynomialRing(CFi);
n:=4096;
pathsastart:=[[[(Floor(n/5)-j)/(Floor(n/5))*0.2*i +
  j/(Floor(n/5))*0.2]: j in [0 .. Floor(n/5)]]];
pathsbstart:=[[[(Floor(n/5)-j)/(Floor(n/5))*0.2*i +
  j/(Floor(n/5))*0.2*i]: j in [0 .. Floor(n/5)]]];
pathscstart:=[[[(Floor(n/5)-j)/(Floor(n/5))*0.2*i +
  j/(Floor(n/5))*(-0.2)]: j in [0 .. Floor(n/5)]]];
pathsa:=[[[1-0.8*Exp(2*Pi(CFi)*i*j/n)]: j in [0 .. n]]];
Append(~(pathsa[1]),[0.2]);
pathsa:=[pathsastart[1] cat pathsa[1] cat Reverse(pathsastart[1])];
pathsdesca:=[[]];
pathsb:=[[[i-0.8*i*Exp(2*Pi(CFi)*i*j/n)]: j in [0 .. n]]];
Append(~(pathsb[1]),[0.2*i]);
pathsb:=[pathsbstart[1] cat pathsb[1] cat Reverse(pathsbstart[1])];
pathsdescb:=[[]];
pathsc:=[[[-1+0.8*Exp(2*Pi(CFi)*i*j/n)]: j in [0 .. n]]];
Append(~(pathsc[1]),[-0.2]);
pathsc:=[pathscstart[1] cat pathsc[1] cat Reverse(pathscstart[1])];
pathsdescc:=[[]];
manypaths:=[pathsa,pathsb,pathsc];
manypathsdesc:=[pathsdesca,pathsdescb,pathsdescc];
liftedmanypaths:=[];liftedmanypathsdesc:=[];
initialize2:=[];initialize3:=[];initialize4:=[];initializelast:=[];
for dmany:=1 to #manypaths do
  paths1:=manypaths[dmany];pathsdesc1:=manypathsdesc[dmany];
  POL<x1>:=PolynomialRing(Pol,1);
  xi1:=POL!(x^4 + (-1/2*x1^8 - x1^4 - 1/2)*x^2 -
    1/48*x1^16 - 1/12*x1^12 - 1/8*x1^8 - 1/12*x1^4 - 1/48);
  paths2,pathsdesc2:=liftpaths(paths1,pathsdesc1,xi1,Pol,initialize2);
  if dmany eq 1 then
    initialize2:=[paths2[l][1][#paths2[l][1]]: l in [1 .. #paths2]];
  end if;
  POL<x1,x2>:=PolynomialRing(Pol,2);
  eta1:=POL!(x^2-(x2^3-(1/2+1/2*x1^4)^2*x2));
```

```
  paths3,pathsdesc3:=liftpaths(paths2,pathsdesc2,eta1,Pol,initialize3);
  if dmany eq 1 then
    initialize3:=[paths3[l][1][#paths3[l][1]]: l in [1 .. #paths3]];
  end if;
  POL<x1,x2,x3>:=PolynomialRing(Pol,3);
  xInWPSp:=POL!(x^2+x2);
  paths4,pathsdesc4:=liftpaths(paths3,pathsdesc3,xInWPSp,Pol,initialize4);
  if dmany eq 1 then
    initialize4:=[paths4[l][1][#paths4[l][1]]: l in [1 .. #paths4]];
  end if;
  POL<x1,x2,x3,x4>:=PolynomialRing(Pol,4);
  yInWPSp:=POL!(x4*x^4-2*x2*x^2-x4^5);
  pathslast,pathsdesclast:=
    liftpaths(paths4,pathsdesc4,yInWPSp,Pol,initializelast);
  if dmany eq 1 then
    initializelast:=
      [pathslast[l][1][#pathslast[l][1]]: l in [1 .. #pathslast]];
  end if;
  Append(~liftedmanypaths,pathslast);
  Append(~liftedmanypathsdesc,pathsdesclast);
end for;

/*We compute a permutation representation from the data just obtained
  and give a matrix representation
*/
G,S,set,gen:=
  permutationrepresentation(liftedmanypaths,liftedmanypathsdesc);
isGCyclicOrder2:=IsIsomorphic(G,CyclicGroup(2));isGCyclicOrder2;
//Give a matrix representation of SL2(F_3)
M:=MatrixGroup<2, FiniteField(3)| [2,0,0,2]>;
rep:=hom<G->M|[M.1,M.1,M.1]>;

/*We do the analog thing for Y_1 that we did for Y_0
*/
Q:=Rationals();
K<t>:=FunctionField(Q);
Pol<x>:=PolynomialRing(K);
JacY1 := EllipticCurve(Pol!x^3-(1/(1+t^4))*x);
g:=Pol!DivisionPolynomial(JacY1,3)/3;g;
K1<xi1>:=ext<K|g>;
Pol1<x1>:=PolynomialRing(K1);
f:=Pol1!g/((x1-xi1)*(x1+xi1));
xi2:=-xi1;
```

```
K2<eta1>:=ext<K1|Pol1!(x1^2-(xi1^3-(1/(1+t^4))*xi1))>;
Pol2<x2>:=PolynomialRing(K2);
K3<eta2>:=ext<K2|Pol2!(x2^2-(xi2^3-(1/(1+t^4))*xi2))>;
Pol3<x3>:=PolynomialRing(K3);
JacY1K3 := EllipticCurve(Pol3!x^3-(1/(1+t^4))*x);
O:=JacY1K3![0,1,0];P1:=JacY1K3![xi1,eta1,1];P2:=JacY1K3![xi2,eta2,1];
zeta:=WeilPairing(P1,P2,3);
Q;K;K1;K2;K3;Degree(K1);Degree(K2);Degree(K3);zeta^2+zeta+1 eq 0;

CFi<i>:=ComplexField(20);
Pol<x>:=PolynomialRing(CFi);
n:=4096;
pathsastart:=[[[(Floor(n/5)-j)/(Floor(n/5))*0.2*i +
j/(Floor(n/5))*0.2]: j in [0 .. Floor(n/5)]]];
pathsbstart:=[[[(Floor(n/5)-j)/(Floor(n/5))*0.2*i +
j/(Floor(n/5))*0.2*i]: j in [0 .. Floor(n/5)]]];
pathscstart:=[[[(Floor(n/5)-j)/(Floor(n/5))*0.2*i +
j/(Floor(n/5))*(-0.2)]: j in [0 .. Floor(n/5)]]];
pathsa:=[[[1-0.8*Exp(2*Pi(CFi)*i*j/n)]: j in [0 .. n]]];
Append(~(pathsa[1]),[0.2]);
pathsa:=[pathsastart[1] cat pathsa[1] cat Reverse(pathsastart[1])];
pathsdesca:=[[]];
pathsb:=[[[i-0.8*i*Exp(2*Pi(CFi)*i*j/n)]: j in [0 .. n]]];
Append(~(pathsb[1]),[0.2*i]);
pathsb:=[pathsbstart[1] cat pathsb[1] cat Reverse(pathsbstart[1])];
pathsdescb:=[[]];
pathsc:=[[[-1+0.8*Exp(2*Pi(CFi)*i*j/n)]: j in [0 .. n]]];
Append(~(pathsc[1]),[-0.2]);
pathsc:=[pathscstart[1] cat pathsc[1] cat Reverse(pathscstart[1])];
pathsdescc:=[[]];
manypaths:=[pathsa,pathsb,pathsc];
manypathsdesc:=[pathsdesca,pathsdescb,pathsdescc];
liftedmanypaths:=[];liftedmanypathsdesc:=[];
initialize2:=[];initialize3:=[];initialize4:=[];initializelast:=[];
for dmany:=1 to #manypaths do
  paths1:=manypaths[dmany];pathsdesc1:=manypathsdesc[dmany];
  POL<x1>:=PolynomialRing(Pol,1);
  xi1:=POL!((x1^4 + 1)^2*x^4 - 2*(x1^4 + 1)*x^2 - 1/3);
  paths2,pathsdesc2:=liftpaths(paths1,pathsdesc1,xi1,Pol,initialize2);
  if dmany eq 1 then
    initialize2:=[paths2[l][1][#paths2[l][1]]: l in [1 .. #paths2]];
  end if;
  POL<x1,x2>:=PolynomialRing(Pol,2);
```

```
  eta1:=POL!((1+x1^4)*x^2-((1+x1^4)*x2^3-x2));
  paths3,pathsdesc3:=liftpaths(paths2,pathsdesc2,eta1,Pol,initialize3);
  if dmany eq 1 then
    initialize3:=[paths3[l][1][#paths3[l][1]]: l in [1 .. #paths3]];
  end if;
  POL<x1,x2,x3>:=PolynomialRing(Pol,3);
  xInWPSp:=POL!((1+x1^4)*x^2-x2);
  paths4,pathsdesc4:=liftpaths(paths3,pathsdesc3,xInWPSp,Pol,initialize4);
  if dmany eq 1 then
    initialize4:=[paths4[l][1][#paths4[l][1]]: l in [1 .. #paths4]];
  end if;
  POL<x1,x2,x3,x4>:=PolynomialRing(Pol,4);
  yInWPSp:=POL!((1+x1^4)*x4*x^4-x2*x^2+(1+x1^4)*x4^5);
  pathslast,pathsdesclast:=
    liftpaths(paths4,pathsdesc4,yInWPSp,Pol,initializelast);
  if dmany eq 1 then
    initializelast:=
      [pathslast[l][1][#pathslast[l][1]]: l in [1 .. #pathslast]];
  end if;
  Append(~liftedmanypaths,pathslast);
  Append(~liftedmanypathsdesc,pathsdesclast);
end for;

G,S,set,gen:=
  permutationrepresentation(liftedmanypaths,liftedmanypathsdesc);
isGCyclicOrder4:=IsIsomorphic(G,CyclicGroup(4));isGCyclicOrder4;
//Give a matrix representation of SL2(F_3)
M:=MatrixGroup<2, FiniteField(3)| [1,1,1,2]>;
rep:=hom<G->M|[M.1,M.1,M.1]>;
```

B.3. Finding Classes in $H^1_{grp}(\pi_1^{an}(A^{an}), \mathrm{Pic}_{Y_i/A}[3])$ with Small Representations

The following script computes small representations of potentially nontrivial Brauer group elements. See section 4.3 for remarks on the underlying theory.

```
/*Using the action found in by the previous scripts of the fundamental
  group "G" of P^1\{4pt} which can be represented as a free group on 3
  generators we determine its cohomology and promissing cocycles with a
  small representation. First we compute the 1-cohomology "H1" using
  MAGMA internal commands and determine the subgroup "Gtrivializing" on
  which the action trivializes.
*/
G<e1,e2,e3>:=Group<e1,e2,e3|>;
```

```
GF3:=GF(3);
Mat:=Matrix(GF3,6,6,[1,1,0,0,0,0,1,2,0,0,0,0,0,0,2,0,0,0,
  0,0,0,2,0,0,0,0,0,0,1,1,0,0,0,0,1,2]);
MGr:=MatrixGroup<6,GF3|Mat>;
N:=[Mat,Mat,Mat];
Gtrivializing:=Kernel(hom<G->MGr|N>);
M:=GModule(G,N);
MM:=CohomologyModule(G,M);
dim:=CohomologicalDimension(MM, 1);
H1:=CohomologyGroup(MM, 1);

/*Now we compute a finite index subgroup over whose quotient all of the
  cohomology in H1 can be represented. For this we take a basis of the
  cohomology group and compute for each basis element which is
  represented as a twisted homomorphism from "G" to the G-module "M" a
  finite index subgroup on which this cocycle always evaluates to 0 as
  discussed in the chapter on examples. This is done by successively
  generating words of small length in "G" and evaluate at these
  elements. There are optimizations applied such as testing only
  elements that are not already generated in small length by the
  already found annihilators and the list of generators of previous
  basis elements is reused to speed up computation. This part could
  probably be replaced by a more direct method to compute a defining
  quotient.
*/
Hlist:=[];Hnlist:=[];
for s in [1..dim] do
  intertestfct:=OneCocycle(MM,H1.s);
  testfct:=function(grpelem)
    return intertestfct(<G!grpelem>);
  end function;
  notenough:=true;
  n:=0;
  gens:=[G|e1,e2,e3];
  B:=[l: l in gens] cat [l^-1: l in gens];
  Hgens:=[G|];H:=sub<G|Hgens>;
  if s gt 1 then
    L:={};
    for i in [1..(s-1)] do
      L:=L join {G!g: g in Generators(Hnlist[i])};
    end for;
    for l in L do
      if testfct(l) eq M!0 then
```

```
        Append(~Hgens,l);
      end if;
    end for;
  end if;
  shortlength:=3;
  while notenough do
    n:=n+1;
    Hgensi:=Hgens cat [l^-1: l in Hgens];Hshort:={G!1};
    for i:=1 to shortlength do
      Hshort:=Hshort join {G!l*b: l in Hshort, b in Hgensi| #(l*b) le n};
    end for;
    if #Hshort gt 1000000 then shortlength:=2; end if;
    L:={G!1};
    for i:=1 to n do
      L:={G!l*b: l in L, b in B| l*b notin Hshort};
    end for;
    print n,#Hgens,#L,#Hshort;
    for l in L do
      if testfct(l) eq M!0 then
        Append(~Hgens,l);
      end if;
    end for;
    H:=sub<G|Hgens>;
    if Index(G,H) ne 0 then
      notenough:=false;
    end if;
  end while;
  Append(~Hnlist,H);
  print s,Index(G,H),#Generators(H),Cputime();
  H:=Core(G,H meet Gtrivializing);
  Append(~Hlist,H);
  print s,Index(G,H),#Generators(H),Cputime();
end for;

/*Now we compute a quotient of definition for the whole "H1" by taking
  the quotient of the normalization of the subgroups generated for each
  basis element above.
*/
F:=G;
for i in [1..dim] do
  F meet:= Hlist[i];
end for;
F:=Core(G,F);
```

```
print Index(G,F),#Generators(F),Cputime();

/*We compute the order of the quotients over which the individual
  basis elements are defined. Since it turns out that there are already
  4 generators with an associated quotient of order 12 by the same
  subgroup we will try to find even smaller quotients of this group
  over which some nontrivial cohomology is defined.
  We also compute the part of these groups of definition which is
  contained in "Gtrivializing". This suggest a splitting of the the
  associated etale extension in one comming from "Gtrivializing" and
  one for the rest.
*/
for i in [1..dim] do
  Index(G,Core(G,Hlist[i]));
end for;
for i in [1..dim] do
  id:=IdentifyGroup(quo<Gtrivializing|Hlist[i]>);
  print id,IsAbelian(SmallGroup(id[1],id[2]));
end for;

/*We print the nontrivial values of the of a cocycle representing one
  of the basis elements with defining group of order 6 on the
  generators of subgroup whose quotient is a group of definition for
  all of the cohomology. Since the group action was split into three
  blocks associated to the description of the Jacobian as isogenous
  to a tripple product of elliptic curve and since all nontrivial
  values are in the 4th collumn it is a reasonable choice to look at
  this factor more closely and neglect the other ones.
*/
intertestfct:=OneCocycle(MM,H1.4);
testfct:=function(grpelem)
  return intertestfct(<G!grpelem>);
end function;
for l in Generators(H) do
  if testfct(l) ne M!0 then print testfct(l); end if;
end for;

/*The quotient of order 12 has a quotient of order 6 isomorphic to the
  symmetric group. We compute the morphism of G to this group, where the
  specifics of different representations have to be dealt with
*/
Q,mapGQ:=quo<G|Core(G,Hlist[4])>;
cosTab:=CosetTable(Q,sub<Q|Id(Q)>);
```

```
phi,QPerm:=CosetTableToRepresentation(Q,cosTab);
Q2,psi:=quo<G|(SubgroupLattice(QPerm)[2]@@phi)@@mapGQ>;
cosTab2:=CosetTable(Q2,sub<Q2|Id(Q2)>);
psi2,QPerm2:=CosetTableToRepresentation(Q2,cosTab2);
_,psi3:=IsIsomorphic(QPerm2,Sym(3));
psi3(psi2(psi(G.1)));
psi3(psi2(psi(G.2)));
psi3(psi2(psi(G.3)));


/*We compute the cohomology of the induced module for the quotient
  "Q2" with 6 elements. The MAGMA implementation is a bit clumsy here
  so we implement this our self. The following functions produce the
  data to supply the MAGMA cohomology machinary with associated to Q2.
*/
compInvSubVec:=function(Gp,Fp,Mp,MVecp)
MInvVecp:=MVecp; idp:=Id(MatrixAlgebra(GF3,Rank(MVecp)));
for i := 1 to NumberOfGenerators(Fp) do
  wordOfFpGen:=Reverse(Seqlist(Eltseq(Gp!Fp.i)));
  matrixelem:=idp;
  while #wordOfFpGen ne 0 do
    l:=wordOfFpGen[#wordOfFpGen];
    if l gt 0 then
      matrixelem:=matrixelem*ActionGenerator(Mp,l);
    else
      matrixelem:=matrixelem*(ActionGenerator(Mp,-l)^(-1));
    end if;
    Prune(~wordOfFpGen);
  end while;
  MInvVecp meet:= Kernel(hom<MVecp->MVecp|matrixelem-idp>);
end for;
return MInvVecp;
end function;

compActMatsOnInvVec:=function(Gp,Mp,MVecp,MInvSubVecp)
monodromyMatrixListRestp:=[]; m:=Rank(MInvSubVecp);
for n := 1 to NumberOfGenerators(Gp) do
  action:=hom<MVecp->MVecp|ActionGenerator(Mp,n)>;
  inducedAct:=hom<MInvSubVecp->MInvSubVecp|
    [action(MInvSubVecp.i): i in [1..m]]>;
  Append(~monodromyMatrixListRestp,Matrix(GF3,m,m,&cat[
    [GF3|Coordinates(MInvSubVecp,inducedAct(MInvSubVecp.i))[j]:
    j in [1..m]]: i in [1..m]]));
```

```
end for;
return monodromyMatrixListRestp;
end function;

/*compQuoGrp comes in two implementations. One which works in general
  and one which uses MAGMA's SmallGroup library. The later has the
  advantage that it gives nicer and much smaller representations for
  the quotient group.
*/
compQuoGrp:=function(G,F)
H,homoepiGToH:=quo<G|F>;
HRed,homoisoHToHRed:=ReduceGenerators(H);
if #HRed in Integers() then
  cosTab:=CosetTable(HRed,sub<HRed|Id(HRed)>);
  _,HPC:=CosetTableToRepresentation(HRed,cosTab);
  HFP,homoisoHFPToHPC:=FPGroup(HPC);
  isofound,homoisoHRedToHFP,homoisoHFPToHRed:=
    SearchForIsomorphism(HRed,HFP,7);
  assert isofound;
  homoepiGToHPC:=hom<G->HPC|[homoisoHFPToHPC(homoisoHRedToHFP(
    homoisoHToHRed(homoepiGToH(g)))): g in Generators(G)]>;
  if Type(HPC) eq GrpPC then HPCGeneratorPreimagelist:=
    [G!(HPC.i@@homoepiGToHPC): i in [1..NumberOfPCGenerators(HPC)]];
    else HPCGeneratorPreimagelist:=[G!(HPC.i@@homoepiGToHPC):
      i in [1..NumberOfGenerators(HPC)]];
  end if;
  return HPC,homoepiGToHPC,HPCGeneratorPreimagelist;
else
  printf "The order of ‘HRed’ in the function ‘compQuoGrp’ is not an
    integer.\n";
  return 0;
end if;
return 0;
end function;

compQuoGrp:=function(G,F)
H,homoepiGToH:=quo<G|F>;
HRed,homoisoHToHRed:=ReduceGenerators(H);
if #HRed in Integers() then
  if CanIdentifyGroup(#HRed) then
    HRedid:=IdentifyGroup(HRed);
    HPC:=SmallGroup(HRedid[1],HRedid[2]);
    HFP,homoisoHFPToHPC:=FPGroup(HPC);
```

```
    isofound,homoisoHRedToHFP,homoisoHFPToHRed:=
      SearchForIsomorphism(HRed,HFP,7);
    assert isofound;
    homoepiGToHPC:=hom<G->HPC|[homoisoHFPToHPC(homoisoHRedToHFP(
      homoisoHToHRed(homoepiGToH(g)))): g in Generators(G)]>;
    if Type(HPC) eq GrpPC then HPCGeneratorPreimagelist:=
      [G!(HPC.i@@homoepiGToHPC): i in [1..NumberOfPCGenerators(HPC)]];
      else HPCGeneratorPreimagelist:=[G!(HPC.i@@homoepiGToHPC):
        i in [1..NumberOfGenerators(HPC)]];
    end if;
    return HPC,homoepiGToHPC,HPCGeneratorPreimagelist;
  else
    printf "Groups of order %o such as ‘HRed’ in the function
      ‘compQuoGrp’ are not covered by Magma’s SmallGroup.\n", #HRed;
    return 0;
  end if;
else
  printf "The order of ‘HRed’ in the function ‘compQuoGrp’ is not an
    integer.\n";
  return 0;
end if;
return 0;
end function;

compActMatsOfQuoOnInvSubVec:=function(HPCGeneratorPreimageListp,
  monodromyMatrixListRestp,MInvSubVecp)
monodromyMatrixListQuop:=[]; m:=Rank(MInvSubVecp);
for i := 1 to #HPCGeneratorPreimageListp do
  wordOfFpGen:=Reverse(Seqlist(Eltseq(HPCGeneratorPreimageListp[i])));
  matrixelem:=Id(MatrixAlgebra(GF3,m));
  while #wordOfFpGen ne 0 do
    l:=wordOfFpGen[#wordOfFpGen];
    if l gt 0 then
      matrixelem:=matrixelem*monodromyMatrixListRestp[l];
    else
      matrixelem:=matrixelem*(monodromyMatrixListRestp[-l]^(-1));
    end if;
    Prune(~wordOfFpGen);
  end while;
  Append(~monodromyMatrixListQuop,matrixelem);
end for;
return monodromyMatrixListQuop;
end function;
```

```
/*The previously computed map to Sym(3) is used.
*/
F:=Kernel(hom<G->Sym(3)|[Sym(3)|(2,3),(1,2),(2,3)]>);

/*We do the cohomology computation for "F" using the functions from
  above. "F" possesses nontrivial cohomology, so it is a candidate to give
  rise to a small representation of a nontrivial Brauer group element. We
  also see that its representing cocycles are nontrivial only on the
  Y0-part of the 3-torsion in the Jacobian.
*/
MVec:=VectorSpace(M);
MInvSubVec:=compInvSubVec(G,F,M,MVec);
monodromyMatrixListSub:=compActMatsOnInvVec(G,M,MVec,MInvSubVec);
Q,epiGToQ,HPCGeneratorPreimageList:=compQuoGrp(G,F);
monodromyMatrixListQ:=compActMatsOfQuoOnInvSubVec(
  HPCGeneratorPreimageList,monodromyMatrixListSub,MInvSubVec);
liftOneCocycle:=function(oncocyclep)
return function(gamma) return &+[oncocyclep(<epiGToQ(gamma[1])>)[i]*
  MInvSubVec.i: i in [1..Rank(MInvSubVec)]]; end function;
end function;
MQ:=GModule(Q,monodromyMatrixListQ);
MMQ:=CohomologyModule(Q,MQ);
dim:=CohomologicalDimension(MMQ, 1);
HQ1:=CohomologyGroup(MMQ, 1);
dim;
[&+[OneCocycle(MMQ,HQ1.1)(<q>)[i]*MInvSubVec.i:
  i in [1..Rank(MInvSubVec)]]: q in Q];
[&+[OneCocycle(MMQ,HQ1.2)(<q>)[i]*MInvSubVec.i:
  i in [1..Rank(MInvSubVec)]]: q in Q];

/*As already mentioned above we try to split the Q2 in a 2-element
  quotient and a 3-element kernel. We compute the kernel of the
  composition of the map from "G" to "Q2" to the group with two
  elements. The generators of that kernel tell us how the loops
  generating the fundamental group lift to the associated degree 2
  cover. From this we derive the covering map u^2=3+t^4 by inspection.
*/
phi:=hom<G->Sym(2)|[Sym(2)|(1,2),(1,2),(1,2)]>;
Kernel(phi);
```

B.4. Finding an Étale Cocycle Representation for Elements of $\mathrm{Br}(X)$

As was mentioned in section 4.1 we know that the fibers of F_0' have j-invariant 1728 and the same holds for the elliptic curve defined by $u^2 = 1 + 3t^4$. We could try to identify the étale 3-cover encoded in the group cohomology data similarly as we identified the cover given by $u^2 = 1 + 3t^4$ in the last section. But it is more complicated in this situation and we already know that we expect the étale covering to be associated to the 3-torsion in the Jacobian of the base curve E. Similarly we suspect, that the second function for the 2-cocycle representative (q_1, q_2) comes from the 3-torsion of the Jacobian of the fibers F_0.

We compute these extensions without further use of the identified group cohomology cocycles. We get several candidates associated to different 3-torsion points. We choose one of them to obtain the obstruction. Actually, we tested several of the candidates, and not all of them gave an obstruction.

Since we want to use 3-torsion points of elliptic curves we need to extend the base field to L the field of definition of the 3-torsion points.

```
/*Set up the base field to split the 3-torsion of the elliptic curve of
  j-invariant 1728. Fix L manually to a slightly simpler form than
  MAGMA gives us.
*/
Q:=Rationals();
E:=EllipticCurve([Q|4,0]);
L:=ext<Q|AbsolutePolynomial(Ring(Universe(PointsOverSplittingField(
  TorsionSubgroupScheme(E,3)))))>;
L:=OptimizedRepresentation(L);L;
tmpPol<tmp>:=PolynomialRing(Q);
L:=ext<Q|tmpPol!tmp^16-6*tmp^12+39*tmp^8+18*tmp^4+9>;

/*Set up an isomorphism between an the F0-fiber and a standard
  representation EL of an elliptic curve with j-invariant 1728
*/
C:=-4;D:=-9;
P2L<u,z,w>:=ProjectiveSpace(L,2);
C1L:=Curve(P2L,u^2*w^2+C*z^4+D*w^4);
_,gamma:=IsPower(L!(-D/C),4);
F1L,n1L:=EllipticCurve(C1L,C1L![0,gamma,1]);
FL:=EllipticCurve([L|4,0]);
n1pL:=Isomorphism(F1L,FL);
n1cL:=n1L*n1pL;
nc:=map<C1L->FL|[f: f in DefiningEquations(n1cL)]>;

/*Compute 3-torsion points and associated divisors. To make up for
  poles of associated functions compute the same for a 2-torsion
```

```
  translate which should give a function with poles at different
  places. Then compute the associated functions and normalize them.
*/
G3:=TorsionSubgroupScheme(FL,3);
Tors3:=RationalPoints(G3);
G2:=TorsionSubgroupScheme(FL,2);
Tors2:=RationalPoints(G2);
Tors3:=Tors3 diff {Id(FL)};
Tors2:=Tors2 diff {Id(FL)};
transl:=Tors2[1];
div1list:=[];
div1listb:=[];
for i in [1..4] do
  point:=Tors3[1];
  Append(~div1list,Divisor(C1L,point@@nc)-Divisor(C1L,(-point)@@nc));
  Append(~div1listb,Divisor(C1L,(transl+point)@@nc)-
    Divisor(C1L,(transl-point)@@nc));
  Tors3:= Tors3 diff {point,-point};
end for;
div1list:=div1list cat div1listb;
f1list:=[];
for i in [1..#div1list] do
  _,f1list[i]:=IsPrincipal(3*div1list[i]);
end for;
for i in [1..#f1list] do
  f1list[i]:=f1list[i]/Evaluate(f1list[i],C1L![0,gamma,1]);
end for;

/*We need to make up for the fact that MAGMA computed in projective
  space while we were pretending they were in weighted projective
  space. This function does the conversion.
*/
convPol:=function(p)
  R:=Parent(p);
  mon:=Monomials(p);
  coef:=Coefficients(p);
  return &+[coef[i]*R.1^a*R.2^b*R.3^(6-2*a-b)
    where a is Exponents(mon[i])[1] where b is Exponents(mon[i])[2]:
    i in [1..#coef]];
end function;

/*Convert the rational functions and check that they are well
  defined modulo the defining equation of the fiber in weighted
```

```
  projective space.
*/
flist:=[];
for i in [1..#f1list] do
  d:=Denominator(ProjectiveFunction(f1list[i]));
  n:=Numerator(ProjectiveFunction(f1list[i]));
  RI:=PreimageRing(Parent(convPol(n)));
  flist[i]:=RI!convPol(n)/RI!convPol(d);
end for;
for i in [1..#flist] do
  f:=flist[i]*Evaluate(flist[i],[-RI.1,RI.2,RI.3]);
  isdiv:=IsDivisibleBy(Numerator(f-1),RI.1^2+C*RI.2^4+D*RI.3^4);
  isdiv;
end for;


/*Do the same thing for the base curve
*/
B:=3;
C2L:=Curve(P2L,u^2*w^2-B*z^4-w^4);
_,gamma:=IsPower(L!(-1/B),4);
E1L,m1L:=EllipticCurve(C2L,C2L![0,gamma,1]);
EL:=EllipticCurve([L|4,0]);
m1pL:=Isomorphism(E1L,EL);
m1cL:=m1L*m1pL;
mc:=map<C2L->EL|[f: f in DefiningEquations(m1cL)]>;

H:=TorsionSubgroupScheme(EL,3);
Tors3:=RationalPoints(H);
Tors3:=Tors3 diff {Id(EL)};
div2list:=[];
for i in [1..4] do
  point:=Tors3[1];
  Append(~div2list,Divisor(C2L,point@@mc)-Divisor(C2L,(-point)@@mc));
  Tors3:= Tors3 diff {point,-point};
end for;
g1list:=[];
for i in [1..4] do
  _,g1list[i]:=IsPrincipal(3*div2list[i]);
end for;
for i in [1..4] do
  g1list[i]:=g1list[i]/Evaluate(g1list[i],C2L![0,gamma,1]);
end for;
```

```
glist:=[];
for i in [1..4] do
  d:=Denominator(ProjectiveFunction(g1list[i]));
  n:=Numerator(ProjectiveFunction(g1list[i]));
  RI:=PreimageRing(Parent(convPol(n)));
  glist[i]:=RI!convPol(n)/RI!convPol(d);
end for;
```

B.5. Local Analysis at 2

We only need to check modulo 2^1 as discussed. So we compute a list of such solutions in an appropriate local ring, evaluate the candidate functions from flist at these points and check that these values are all 3rd powers.

```
/*Define a function to evaluate rational functions at given 2-adic points.
*/
evalPolViaMap:=function(p,point,map)
  mon:=Monomials(p);
  coef:=Coefficients(p);
  exp:=[Exponents(mon[i]): i in [1..#coef]];
  eins:=Codomain(map)!1;
  return &+[map(coef[i])*
    ((exp[i][1] ne 0) select point[1]^exp[i][1] else eins)*
    ((exp[i][2] ne 0) select point[2]^exp[i][2] else eins)*
    ((exp[i][3] ne 0) select point[3]^exp[i][3] else eins):
    i in [1..#coef]];
end function;

/*Initialize the rings needed for the 2-adic computations.
*/
O:=Integers(L);
L2,mLL2:=Completion(L,Decomposition(O,2)[1][1]:Precision:=100);
O2b<a2>:=Completion(O,Decomposition(O,2)[1][1]:Precision:=100);
B2b:=BaseRing(O2b);
AssignNames(~B2b,["b2"]);
B2:=BaseRing(B2b);

/*Compute all 2-adic solutions up +/- for x3 up to the precision given
  by "powbound".
*/
Quad:=[];
powbound:=4;
for x0 in [1 .. (2^(powbound  ))-1] do
```

```
  for x1 in [0 .. (2^(powbound  ))-1] do
    for x2 in [0 .. (2^(powbound-2))-1] do
      if GCD([x0,x1,x2]) eq 1 then
        isPartOfSol,x3:=IsPower(((B2!x0)^4+3*(B2!x1)^4-4*(B2!x2)^4)/9,4);
        if isPartOfSol then
           Append(~Quad,[B2!x0,B2!x1,B2!x2,B2!x3]);
        end if;
      end if;
    end for;
  end for;
end for;
for x1 in [1 .. (2^(powbound  ))-1] do
  for x2 in [0 .. (2^(powbound-2))-1] do
    if GCD([x1,x2]) eq 1 then
      isPartOfSol,x3:=IsPower(((B2!0)^4+3*(B2!x1)^4-4*(B2!x2)^4)/9,4);
      if isPartOfSol then
        Append(~Quad,[B2!0,B2!x1,B2!x2,B2!x3]);
      end if;
    end if;
  end for;
end for;
for x2 in [1.. (2^(powbound-2))-1] do
  isPartOfSol,x3:=IsPower(((B2!0)^4+3*(B2!0)^4-4*(B2!x2)^4)/9,4);
  if isPartOfSol then
    Append(~Quad,[B2!0,B2!0,B2!x2,B2!x3]);
  end if;
end for;
#Quad;

/*Check if q_2 evaluates to a 3rd power at all truncated 2-adic solutions.
*/
f:=flist[3];
for s in Quad do
  _,u:=IsPower(L2!(s[1]^4+3*s[2]^4),2);
  pt:=[u*s[1]^2,s[3],s[4]];
  d:=evalPolViaMap(Denominator(f),pt,mLL2);
  if IsUnit(d) then
    n:=evalPolViaMap(Numerator(f),pt,mLL2);
    bool:=IsPower(n/d,3);
    if not bool then pt; end if;
  end if;
end for;
```

B.6. Local Analysis at 3

These are the scripts, that realize the computations explained at the end of section 4.1.

```
/*Initialize the rings needed for the 2-adic computations.
*/
O:=Integers(L);
L3,mLL3:=Completion(L,Decomposition(O,3)[1][1]:Precision:=500);
M<zeta>:=UnramifiedExtension(L3,3);
PM<xM>:=PolynomialRing(M);
zeta3:=-Coefficients(Factorisation(PM!(xM^3-1))[2][1])[1];
K3:=BaseField(L3);
k3:=BaseField(K3);
Z3:=RingOfIntegers(k3);
FF3,mFF3:=ResidueClassField(Z3);
LauK3<x>:=LaurentSeriesRing(K3);
O3b<a3>,mapOToO3b:=Completion(O,Decomposition(O,3)[1][1]:Precision:=1000);
B3b:=BaseRing(O3b);
AssignNames(~B3b,["b3"]);
B3:=BaseRing(B3b);

/*The following functions follow the description in Neukirch p.355/356,
  and are needed as auxiliary functions to compute the local invariant
  at 3.
*/
expand:=function(l,unif)
  v:=Valuation(l);
  f:=l;
  Fi:=Parent(unif);
  R:=RingOfIntegers(Fi);
  Q,m:=ResidueClassField(R);
  r:=Fi!(m(f)@@m);
  return r, l-r;
end function;

itexpand:=function(l,unif,n)
  rvec:=[];
  v:=Valuation(l);
  rest:=l*unif^(-v);
  for i in [1..n] do
    r,rest:=expand(rest,unif);
    rest:=rest/unif;
    Append(~rvec,r);
```

```
  end for;
  return rvec, rest*unif^(n+v);
end function;

laurentify:=function(l,Lau,Ext,n)
  Fi:=BaseRing(Lau);
  v:=Valuation(l);
  list:=itexpand(l,UniformizingElement(Ext),n);
  return &+[Fi!(list[i])*Lau.1^(v+i-1): i in [1..n]];
end function;

frobOp:=function(ser)
  Lau:=Parent(ser);
  coef,exp:=Coefficients(ser);
  char:=Characteristic(ResidueClassField(RingOfIntegers(BaseRing(Lau))));
  nser:=&+[GaloisImage(coef[i],1)*Lau.1^(char*(exp+i-1)): i in [1..#coef]];
  return nser;
end function;

log:=function(l,n)
  list:=[l-1];
  for i in [2..n] do
    Append(~list,list[#list]*(l-1)+BigO(Parent(l).1^n));
  end for;
  return &+[(-1)^(i-1)*list[i]/i: i in [1..n]];
end function;

dlog:=function(l)
  return Derivative(l)/l;
end function;

hfunct:=function(zeta,Lau,Ext,n,nth)
  s:=laurentify(zeta,Lau,Ext,n);
  inter:=s^(Characteristic(ResidueClassField(
    RingOfIntegers(BaseRing(Lau))))^nth)-1;
  coef,v:=Coefficients(inter);
  notfound:=true; vintoffset:=0;
  while notfound do
    vintoffset:=vintoffset+1;
    if (Valuation(coef[vintoffset]) eq 0) or (vintoffset eq #coef) then
      notfound:=false;
    end if;
  end while;
```

```
  if vintoffset eq #coef then
    print "Precission is not be enough!";
  end if;
  vint:=vintoffset+v-1;
  coefint:=coef[vintoffset];
  t:=[];
  for i in [1..#coef] do
    if (not IsWeaklyZero(coef[i])) and (i ne vintoffset) then
      Append(~t,(coef[i]/coefint)*Lau.1^(i-vintoffset));
    end if;
  end for;
  t:=&+t;
  list:=[t];
  nth:=n;
  /*Precision v*(vint-truevaluation+1) would be enough, where
    truevaluation is the first non-weakly-zero-coefficient.
  */
  for i in [2..2*(nth+2)] do
    Append(~list,list[#list]*t+BigO(Lau.1^(2*n*(vint+1))));
  end for;
  tinv:=(Lau.1^(-vint)/coefint)*(Lau!1+(&+[(-1)^i*list[i]:
    i in [1..2*(nth+1)]]));
  return 1/2+tinv;
end function;

/*We set up the function "h" from Neukirch p.355/356, as well as a
  description of L_3^*/L_3^(3).
*/
p:=3;
prec:=40;
h:=hfunct(zeta3,LauK3,L3,prec,1);
coef,v:=Coefficients(h);
h:=&+[coef[i]*LauK3.1^(v+i-1): i in [-Valuation(h)-20+1..#coef]];
G,mG:=pSelmerGroup(3,L3);

/*Compute all 3-adic solutions up +/- for x2 up to the precision given
  by "powbound". Then we check the u^3-criterion.
*/
Quad:=[];
powbound:=4;
for x0 in [1 .. (3^(powbound  ))-1] do
  for x1 in [0 .. (3^(powbound-1))-1] do
    for x3 in [0 .. (3^(powbound-2))-1] do
```

```
      if GCD([x0,x1,x3]) eq 1 then
        isPartOfSol,x2:=IsPower(((B3!x0)^4+3*(B3!x1)^4-9*(B3!x3)^4)/4,4);
        if isPartOfSol then
          Append(~Quad,[B3!x0,B3!x1,B3!x2,B3!x3]);
        end if;
      end if;
    end for;
  end for;
end for;
for x1 in [1 .. (3^(powbound-1))-1] do
  for x3 in [0 .. (3^(powbound-2))-1] do
    if GCD([x1,x3]) eq 1 then
      isPartOfSol,x2:=IsPower(((B3!0)^4+3*(B3!x1)^4-9*(B3!x3)^4)/4,4);
      if isPartOfSol then
        Append(~Quad,[B3!0,B3!x1,B3!x2,B3!x3]);
      end if;
    end if;
  end for;
end for;
for x3 in [1 .. (3^(powbound-2))-1] do
  isPartOfSol,x2:=IsPower(((B3!0)^4+3*(B3!0)^4-9*(B3!x3)^4)/4,4);
  if isPartOfSol then
    Append(~Quad,[B3!0,B3!0,B3!x2,B3!x3]);
  end if;
end for;
for s in Quad do
  _,u:=IsPower(s[1]^4+3*s[2]^4,2);
  if Valuation(u) ne 0 then print index, Valuation(u), s; end if;
end for;

/*Compute all 3-adic solutions up to the precision given by "powbound".
*/
Quad:=[];
powbound:=4;
for x0 in [1 .. (3^(powbound  ))-1] do
  for x1 in [0 .. (3^(powbound-1))-1] do
    for x3 in [0 .. (3^(powbound-2))-1] do
      if GCD([x0,x1,x3]) eq 1 then
        isPartOfSol,x2:=IsPower(((B3!x0)^4+3*(B3!x1)^4-9*(B3!x3)^4)/4,4);
        if isPartOfSol then
          Append(~Quad,[B3!x0,B3!x1,B3!x2,B3!x3]);
          Append(~Quad,[B3!x0,B3!x1,-B3!x2,B3!x3]);
        end if;
```

```
      end if;
    end for;
  end for;
end for;
for x1 in [1 .. (3^(powbound-1))-1] do
  for x3 in [0 .. (3^(powbound-2))-1] do
    if GCD([x1,x3]) eq 1 then
      isPartOfSol,x2:=IsPower(((B3!0)^4+3*(B3!x1)^4-9*(B3!x3)^4)/4,4);
      if isPartOfSol then
        Append(~Quad,[B3!0,B3!x1,B3!x2,B3!x3]);
        Append(~Quad,[B3!0,B3!x1,-B3!x2,B3!x3]);
      end if;
    end if;
  end for;
end for;
for x3 in [1 .. (3^(powbound-2))-1] do
  isPartOfSol,x2:=IsPower(((B3!0)^4+3*(B3!0)^4-9*(B3!x3)^4)/4,4);
  if isPartOfSol then
    Append(~Quad,[B3!0,B3!0,B3!x2,B3!x3]);
    Append(~Quad,[B3!0,B3!0,-B3!x2,B3!x3]);
  end if;
end for;

/*This function evaluation rational functions at a specified p-adic point.
*/
evalPolViaMap:=function(p,point,map)
  mon:=Monomials(p);
  coef:=Coefficients(p);
  exp:=[Exponents(mon[i]): i in [1..#coef]];
  eins:=Codomain(map)!1;
  return &+[map(coef[i])*
    ((exp[i][1] ne 0) select point[1]^exp[i][1] else eins)*
    ((exp[i][2] ne 0) select point[2]^exp[i][2] else eins)*
    ((exp[i][3] ne 0) select point[3]^exp[i][3] else eins):
    i in [1..#coef]];
end function;

/*Compute the local invariants for (q_1,q_2) evaluated at the truncated
  3-adic solutions as well as the mod 3 part of x_1x_3 and store it.
  This data confirms that (q_1,q_2) represents an obstruction to weak
  approximation.
*/
prodinvlist:=[[0,0,0],[0,0,0],[0,0,0]];
```

```
fct1:=glist[1];
fct2:=flist[3];
for s in Quad do
  vals:=[];
  _,u:=IsPower(s[1]^4+3*s[2]^4,2);
  pt:=[u,s[2],s[1]];
  n:=evalPolViaMap(Numerator(fct1),pt,mLL3);
  d:=evalPolViaMap(Denominator(fct1),pt,mLL3);
  Append(~vals,n/d);
  pt:=[u*s[1]^2,s[3],s[4]];
  n2:=evalPolViaMap(Numerator(fct2),pt,mLL3);
  d2:=evalPolViaMap(Denominator(fct2),pt,mLL3);
  Append(~vals,n2/d2);
  fb:=ChangePrecision(vals[1],537);
  gb:=ChangePrecision(vals[2],537);
  f:=laurentify(fb,LauK3,L3,prec);
  g:=laurentify(gb,LauK3,L3,prec);
  srep:=h*((1/p)*log(g^p/frobOp(g),prec)*dlog(f) -
    (1/p)*log(f^p/frobOp(f),prec)*(1/p)*dlog(frobOp(g)));
  slist,sv:=Coefficients(srep);
  if sv lt 0 then res:=slist[-sv]; else res:=K3!0; end if;
  inv:=mFF3(Z3!Trace(res));
  prod:=mFF3(Z3!(s[2]*s[4]));
  prodinvlist[Integers()!prod+1][Integers()!inv+1]:=
    prodinvlist[Integers()!prod+1][Integers()!inv+1]+1;
  if (inv eq 0 and prod ne 0) then print prod, inv, s; end if;
end for;
print #Quad, prodinvlist;
```

B.7. Code for an Alternative Fibration

The following script computes explicit equations and the geometric genus for the covering of the base curve of the alternative fibration, over which the 3-torsion subgroup of the Jacobian is trivialized. At first we compute defining equations for the extensions of the generic point of the base curve using the function field arithmetic functionality of MAGMA. The extensions are derived in several steps to get the splitting field of the division polynomial for 3-torsion points of an elliptic curve. Then we use these equations to define successively affine non-smooth models of the extensions, and desingularize and projectivize the models using point blowups and normaliztion implemented in MAGMA.

For the computation of the field extension we choose an order for adjoin roots. We get a tower of field extensions $K_4/K_3/K_2/K_1/\mathbb{Q}$. The extensions $K_3/K_2/K_1$ are essentially interchangeable. This order is more convenient for computing the field extension tower ,but we reverse this order for the computation of extension of the actual curves. The extension K_4/K_3 consists in adjoining a 3rd root of unity ζ_3, which we do at the start of the computations for the curves.

The computations are not entirely self contained. E.g., MAGMA's Blowup-command returns two affine charts, and one has to choose a suitable one. Also the equations for the morphism have to be given manually, but this is not a big problem since they are only quadratic transforms.

At various instances the equations given by a previous MAGMA-command are in an incompatible format with following computations. So at several instances, we manually insert equations, rather than extracting them tediously by internal commands. To give an example: $a_{2,1}$ is a polynomial in $\mathbb{Q}(\zeta_3, u, v)[z]$:

$$z^2 + \frac{-u^8/32 + u^7v/8 - 3u^6v^2/16 + u^5v^3/8 - u^4v^4/32}{u^5v^3 - 5u^4v^4/2 + 5u^3v^5/2 - 5u^2v^6/4 + 5uv^7/16 - v^8/32}.$$

MAGMA's functionality with polynomials over function fields is limited, thus we dehomogenize it by setting $v = 1$, and convert it to $a_{2,2,norm} \in \mathbb{Q}(\zeta_3)[u, z]$:

$$(u^5 - 5u^4/2 + 5u^3/2 - 5u^2/4 + 5u/16 - 1/32)z^2 \ + \ (u^8/32 - u^7/8 + 3u^6/16 - u^5/8 + u^4/32).$$

We use blowups in the situation when the singular points are easy, because MAGMA computes them faster than normalizations. The Blowup-command performs the blow up at a fixed center in the affine plane so we need to translate the singularities to that center. Especially at the end this gets to tedious, so we apply normalization at the expense of computational overhead.

We get the defining equation $a^4 - ab^3 - 9c^4$ for the completion BC of B and also a morphism $\chi' : BC \to \mathbb{P}^1$.

```
/*Determine a field of definition K4 for 3-torsion of the elliptic
  curve over the generic point
*/
Q:=Rationals();
K<C>:=FunctionField(Q);
Pol<x>:=PolynomialRing(K);
E3 := EllipticCurve(Pol!x*(x+1)*(x+C));
g:=Pol!DivisionPolynomial(E3,3)/3;//g;
K1<xi1>:=ext<K|g>;
Pol1<x1>:=PolynomialRing(K1);
//Pol1!g/(x1-xi1);
K2<xi2>:=ext<K1|Pol1!g/(x1-xi1)>;
Pol2<x2>:=PolynomialRing(K2);
K3<eta1>:=ext<K2|Pol2!(x2^2-(xi1*(xi1+1)*(xi1+C)))>;
Pol3<x3>:=PolynomialRing(K3);
K4<eta2>:=FunctionField(Pol3!(x3^2-(xi2*(xi2+1)*(xi2+C))):
  Check:=false);
Pol4<x4>:=PolynomialRing(K4);
E0 := EllipticCurve(Pol4!x4*(x4+1)*(x4+C));
```

```
O:=E0![0,1,0];P1:=E0![xi1,eta1,1];P2:=E0![xi2,eta2,1];
zeta:=WeilPairing(P1,P2,3);
Q;K;K1;K2;K3;K4;Degree(K1);Degree(K2);Degree(K3);Degree(K4);
zeta^2+zeta+1 eq 0;

/*Set the base field to contain a 3rd root of unity.
*/
Qz<zeta>:=CyclotomicField(3);
PolQz<u>:=PolynomialRing(Qz);

/*Compute the cover of curves related to K1(zeta)/Qz.
*/
P1<C,var>:=ProjectiveSpace(Qz,1);
A<C,x>:=AffineSpace(Qz,2);
Polz:=CoordinateRing(A);
DefiningPolynomial(K1);
a11:=Polz!(x^4 + (4/3*C + 4/3)*x^3 + 2*C*x^2 - 1/3*C^2);
Discriminant(g);
B11:=Curve(A,a11);
B11p<a,b,c>:=ProjectiveClosure(B11);
projB11p:=map<B11p->P1|[a,c]>;
SingularPoints(B11p);
_,B12:=Blowup(B11);
B12p:=ProjectiveClosure(B12);
f11:=map<B12p->B11p|[a*b,b*c,c^2]>;
SingularPoints(B12p);
transB12:=Translation(A,A![-1,-1]);
B13:=Blowup(transB12(B12));
B13p:=ProjectiveClosure(B13);
f12:=map<B13p->B12p|[(a-c)*c,a*b-c^2,c^2]>;
SingularPoints(B13p);
B13H<x,var>:=AffinePatch(B13p,3);
B13Hp:=ProjectiveClosure(B13H);
_,B14:=Blowup(B13H);
B14p:=ProjectiveClosure(B14);
f13:=map<B14p->B13p|[a^2,b*c,a*c]>;
SingularPoints(B14p);
_,B15:=Blowup(AffinePatch(B14p,1));
B15p:=ProjectiveClosure(B15);
f14:=map<B15p->B14p|[a*b,b*c,c^2]>;
SingularPoints(B15p);
f1pre:=Extend(f14*f13*f12*f11);
f1tmp:=Extend(f1pre*projB11p);
```

```
RationalPoints((P1![0,1])@@f1tmp);
RationalPoints((P1![1,1])@@f1tmp);
RationalPoints((P1![1,0])@@f1tmp);
_,B16p:=IsConic(B15p);
B1p<u,v>:=ProjectiveSpace(Qz,1);
Parametrization(B16p, Curve(B1p));
f15:=map<B1p->B15p|[v^2,u^2,2*u*v-v^2]>;
f1:=Extend(f15*f1tmp);
f1des:=Extend(f15*f1pre);

/*Compute the cover of curves related to K3(zeta)/K2(zeta).
*/
FFAux<u,v>:=FunctionField(Qz,2);
PolAux<z>:=PolynomialRing(FFAux);
CAux:=(u-v)^3*(3*u-v)/(v*(v-2*u)^3);
xAux:=(u-v)^2/(v*(2*u-v));
DefiningPolynomial(K3);
a21:=PolAux!(z^2-xAux*(xAux+1)*(xAux+CAux));
a21*(u-1/2*v)^5*v^3;
Polz<u,v>:=PolynomialRing(Qz,2);
Numerator(Discriminant(a21));
discrf2num:=Polz!(-1/8*u^8+1/2*u^7*v-3/4*u^6*v^2+1/2*u^5*v^3-1/8*u^4*v^4);
Factorisation(discrf2num);
A<u,z>:=AffineSpace(Qz,2);
Polz:=CoordinateRing(A);
a22norm:=Polz!((u^5*1^3-5/2*u^4*1^4+5/2*u^3*1^5-5/4*u^2*1^6+
  5/16*u*1^7-1/32*1^8)*z^2 +
  1/32*u^8-1/8*u^7*1+3/16*u^6*1^2-1/8*u^5*1^3+1/32*u^4*1^4);
B22:=Curve(A,a22norm);
B22p<a,b,c>:=ProjectiveClosure(B22);
projB22p:=map<B22p->B1p|[a,c]>;
SingularPoints(B22p);
L:=Normalisation(Ideal(a22norm));
IB22N:=L[1][1];
PolN<h1,h2>:=Generic(IB22N);
mapPolzToPolN:=L[1][2];
B22N:=Curve(Spec(quo<PolN|IB22N>));
f22Naff:=map<B22N->B22|[mapPolzToPolN(u),mapPolzToPolN(v)]>;
B22Np<a,b,c>:=ProjectiveClosure(B22N);
B22NptsAtInf:=B22Np meet Scheme(Ambient(B22Np),c);
Spts:=RationalPoints(B22NptsAtInf);
IsSingular(B22Np,B22Np![7/2,1,0]);
IsSingular(B22Np,B22Np![1,0,0]);
```

```
/*The points at infinity are therefor nonsingular and the others are
  due to normalization.
*/
f22N:=ProjectiveClosure(f22Naff);
f2des:=Extend(f22N*projB22p);
_,B23p:=IsConic(B22Np);
B2p<u,v>:=ProjectiveSpace(Qz,1);
Parametrization(B23p, Curve(B2p));
f23:=map<B2p->B22Np|[-7/2*u^2 + 1/64*v^2,-u^2,u*v]>;
f2:=Extend(f23*f2des);
f21des:=map<B2p->B11p|[3*u^8+1/2*u^6*v^2+3/128*u^4*v^4-1/65536*v^8,
  -4*u^8-1/2*u^6*v^2-1/64*u^4*v^4,u^6*v^2]>;
f21des eq Expand(f2*f1des);

/*Compute the cover of curves related to K2(zeta)/K1(zeta).
*/
FFAux<u,v>:=FunctionField(Qz,2);
PolAux<y>:=PolynomialRing(FFAux);
CAux:=(3*u^8 + 1/2*u^6*v^2 + 3/128*u^4*v^4 - 1/65536*v^8)/(u^6*v^2);
xAux:=(-4*u^8 - 1/2*u^6*v^2 - 1/64*u^4*v^4)/(u^6*v^2);
DefiningPolynomial(K2);
a31:=PolAux!(y^3+(xAux+(4/3*CAux+4/3))*y^2+
  (xAux^2+(4/3*CAux+4/3)*xAux+2*CAux)*y+xAux^3+
  (4/3*CAux+4/3)*xAux^2+2*CAux*xAux);
a31*u^10*v^2;
Polz<u,v>:=PolynomialRing(Qz,2);
Numerator(Discriminant(a31));
discrf2num:=Polz!(-243/16*u^32 + 135/512*u^28*v^4 - 477/262144*u^24*v^8 +
  635/100663296*u^20*v^12 - 10643/927712935936*u^16*v^16 +
  635/59373627899904*u^12*v^20 - 53/10133099161583616*u^8*v^24 +
  5/389111007804810 8544*u^4*v^28 - 1/79689934398425262981 12*v^32);
Factorisation(discrf2num);
A<u,y>:=AffineSpace(Qz,2);
Polz:=CoordinateRing(A);
a32norm:=Polz!(u^10*1^2*y^3 +
  (3/2*u^10*1^2 + 1/64*u^8*1^4 - 1/49152*u^4*1^8)*y^2 +
  (3/16*u^10*1^2 + 1/64*u^8*1^4 - 1/6144*u^6*1^6 -
  1/49152*u^4*1^8 + 1/3145728*u^2*1^10)*y +
  (-3/4*u^12 - 5/32*u^10*1^2 - 31/3072*u^8*1^4 - 1/12288*u^6*1^6 +
  3/262144*u^4*1^8 + 1/6291456*u^2*1^10 - 1/201326592*1^12));
B32:=Curve(A,a32norm);
B32p<a,b,c>:=ProjectiveClosure(B32);
projB32p:=map<B32p->B2p|[a,c]>;
```

```
SingularPoints(B32p);
L:=Normalisation(Ideal(a32norm));
IB32N:=L[1][1];
PolN<h1,h2>:=Generic(IB32N);
mapPolzToPolN:=L[1][2];
B32N:=Curve(Spec(quo<PolN|IB32N>));
f32Naff:=map<B32N->B32|[mapPolzToPolN(u),mapPolzToPolN(v)]>;
B32Np<a,b,c>:=ProjectiveClosure(B32N);
f32N:=ProjectiveClosure(f32Naff);
B32NptsAtInf:=B32Np meet Scheme(Ambient(B32Np),c);
RationalPoints(B32NptsAtInf);
f3:=Extend(f32N*projB32p);
f321:=Extend(f3*f21des*projB11p);
P1C:=Curve(P1);
/*MAGMA could do the computations but it gives unnecessary
  complicated equations:
  fCpre:=map<B32Np->P1C|[DefiningEquations(f321)[1],
    DefiningEquations(f321)[2]]>;
*/
fCpre:=Extend(map<B32Np->P1C|[
  6697777828104831776349602398079493706088448*a^24 -
  2260500016985380724517990809351829125804 8512*a^21*b^3 +
  3337769556329976226046095804433560193571 2256*a^18*b^6 -
  2816243063153417440726393334990816413325 7216*a^15*b^9 +
  1485128177834809978508058985249063342964 7360*a^12*b^12 -
  501230760019248367746469907521558878250 5984*a^9*b^15 +
  105728363441560202571520996117828825880 9856*a^6*b^18 -
  127440438076880601313886914963454388338688*a^3*b^21 +
  6720491851710500459912005281275915010048*b^24 +
  1/2*a^18*c^6 - 81/64*a^15*b^3*c^6 + 10935/8192*a^12*b^6*c^6 -
  98415/131072*a^9*b^9*c^6 + 7971615/33554432*a^6*b^12*c^6 -
  43046721/1073741824*a^3*b^15*c^6 + 387420489/137438953472*b^18*c^6 +
  1/9525728466637982970808323410601946604214 6816*a^12*c^12 -
  1/5644876128378063982701228687764116506201 2928*a^9*b^3*c^12 +
  1/8920298079412249256614287309059344602392 1664*a^6*b^6*c^12 -
  9/2854495385411919762116571938898990272765 493248*a^3*b^9*c^12
  + 243/7307508186654514591018424163581415098279 66271488*b^12*c^12 -
  1/729303636146204352010738683026811956002877829637858210319909678313 38
  0823521771348975037043588324820676234643521860704918238297849856*c^24,
  a^18*c^6 - 81/32*a^15*b^3*c^6 + 10935/4096*a^12*b^6*c^6 -
  98415/65536*a^9*b^9*c^6 + 7971615/16777216*a^6*b^12*c^6 -
  43046721/536870912*a^3*b^15*c^6 + 387420489/68719476736*b^18*c^6]>);
```

```
/*Define the completion of B and combine all the morphism. Compute
  preimages of the ramification points and the geometric genus.
*/
BC:=Curve(Ambient(B32Np),a^4-a*b^3-9*c^4);
fC:=Extend(map<BC->B32Np|[a,4/3*b,(2^22*3^2)*c]>*fCpre);
#RationalPoints((P1C![0,1])@@fC,ext<Qz|PolQz.1^2+1>);
#RationalPoints((P1C![1,1])@@fC);
#RationalPoints((P1C![1,0])@@fC);
Genus(BC);
```

APPENDIX C

A List of Small Solutions to $x_0^4 + 3x_1^4 - 4x_2^4 - 9x_3^4 = 0$

We computed all primitive integral solutions of $x_0^4 + 3x_1^4 - 4x_2^4 - 9x_3^4 = 0$ which satisfy $x_0^4+3x_1^4 \leq (1+3)\cdot(500000)^4$. In particular it contains all solutions with each coordinate below 500000. The quadruples are ordered by the value of $x_0^4 + 3x_1^4$.

```
[ 1, 1, 1, 0 ],
[ 7, 3, 5, 2 ],
[ 95, 63, 73, 36 ],
[ 43, 167, 155, 42 ],
[ 101, 201, 157, 130 ],
[ 331, 393, 103, 310 ],
[ 715, 39, 307, 398 ],
[ 299, 581, 65, 444 ],
[ 767, 273, 85, 448 ],
[ 1189, 1173, 911, 860 ],
[ 2509, 1923, 2119, 160 ],
[ 2093, 2847, 305, 2214 ],
[ 8015, 8193, 5147, 6372 ],
[ 15521, 43, 10975, 318 ],
[ 12083, 21069, 5869, 16120 ],
[ 29281, 5121, 20665, 5408 ],
[ 29749, 2491, 20119, 10920 ],
[ 30133, 14421, 8525, 17944 ],
[ 16267, 49671, 35515, 33958 ],
[ 67673, 7299, 24637, 38370 ],
[ 13247, 56847, 50947, 26460 ],
[ 35485, 74013, 10579, 56476 ],
[ 5087, 84831, 30065, 64116 ],
[ 115309, 55309, 39691, 68220 ],
[ 27983, 95343, 8825, 72488 ],
[ 148589, 10287, 35453, 85510 ],
[ 135377, 179797, 54395, 139782 ],
[ 108827, 202187, 188729, 54420 ],
[ 39047, 214311, 199457, 7000 ],
[ 302453, 80169, 212335, 79586 ],
[ 398011, 74439, 143695, 226006 ],
[ 494999, 80777, 255329, 263160 ],
[ 528469, 327243, 178429, 331240 ].
```

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